The Fundamentals of Mathematical Analysis Volume 1

G.M.Fikhtengol'ts

Translation edited by I. N. Sneddon, Professor of Mathematics in the University of Glasgow

THE FUNDAMENTALS OF MATHEMATICAL ANALYSIS

Volume I

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FOREWORD

THIS book is planned as a textbook of analysis for first and second year mathematics students at Russian universities and consequently is divided into two volumes. In compiling the book I have made extensive use of my three-volume *Course of Differential and Integral Calculus*, revising and abridging it in order to adapt it to the official mathematical analysis programme and to make it meet the requirements of a lecture course.

The tasks I set myself and the points by which I was guided are as follows:

1. First and foremost to provide a systematic and, as far as possible, rigorous treatment of the fundamentals of mathematical analysis. I consider it obligatory for the contents of a textbook to be presented in a logical sequence, in order to achieve a clearly defined and systematic presentation of the facts.

This does not, however, prevent the lecturer from deviating from a strict systematic approach, but, perhaps, even helps him in this respect. In my own lecture courses, for example, I usually put aside for a while such difficult tasks for beginners as the theory of real numbers, the principle of convergence or the properties of continuous functions.

2. To uphold my own opinion that a course of mathematical analysis should not appear to students to be merely a long chain of "definitions" and "theorems", but that it should also serve as a guide to action. Students must be taught to apply the theorems in practice in order to assist them in mastering the computational apparatus of analysis. Although this can be achieved largely with the help of exercises, I have also included some examples in my treatment of the theoretical material. The total number of these examples is, out of necessity, small, but they have been selected in such a way as to prepare students for conscientious work on the exercises.

3. It is well known that mathematical analysis has diverse and remarkable applications both in mathematics itself and in related scientific fields. Whilst students will realize this more and more as time passes, it is essential that they should learn and get used to the relationship of mathematical analysis with other mathematical sciences and with the requirements of practical work whilst studying the fundamentals of analysis. For this very reason I have provided, wherever possible, examples of the application of analysis not only to geometry, but also to mechanics, physics and engineering.

4. The problem of completing analytic work up to numerical results is of both theoretical and practical importance. Since an "exact" or "closed form" solution of a problem in analysis is possible in the simplest cases only, it is important to acquaint students with the use of approximate methods. Some attention has been given to this within the pages of this book.

5. By way of a brief explanation of my treatment of the subject matter, I have first of all considered the concept of a limit which plays the principal role among the fundamental concepts of analysis and which crops up in diverse forms literally throughout the entire course. Hence arises the problem of establishing a unified form of all variations of the limit. This is not only important from the viewpoint of principles but also vital from a practical standpoint, to obviate the necessity of having to construct the theory of limits anew each time it arises. There are two ways of achieving this aim: we can either immediately give the general definition of the limit of "directed variable" (following, for example, Shatunovskii and Moore, or Smith), or we can reduce every limit to the simplest case of the limit of a variable ranging over an enumerated sequence of values. The first alternative is difficult for beginners, and I have, therefore, chosen the second method of approaching the problem. The definition of each new limit is given first by means of the limit of a sequence and only later on "in ε–δ language".

6. To indicate a second feature of my treatment of the subject matter I have in Volume II, when speaking of curvilinear and surface integrals, emphasized the difference between the curvilinear and surface "integrals of first kind" (the exact counterparts of the

ordinary and double integral over unoriented domains) and similar "integrals of second kind" (where the analogy partly vanishes). Experience has convinced me that this distinction not only leads to a better understanding of the material, but is also convenient in applications.

7. As a short appendix to the book I have included a brief account of elliptic integrals and in several cases I have presented problems with solutions involving elliptic integrals. This may help to destroy the harmful illusion, acquired by merely solving simple problems, that the results of analytic calculations must necessarily be "elementary".

8. In various places throughout the book the reader will come across remarks of an historical nature. Moreover, Volume I ends with a chapter entitled, "Historical survey of the development of the fundamental concepts of mathematical analysis" and Volume II concludes with "An outline of further developments in mathematical analysis". However, neither of these two "surveys" has been introduced to serve as a substitute for a complete history of mathematical analysis, which students meet with later in general courses on "the history of mathematics". The first survey touches upon the origin of the concepts, whilst the final chapter in Volume II aims at providing the reader with at least a general *idea of the chronology* of the most important events in the history of analysis.

At this point, and in connection with the preceding paragraph, I should like to give a warning to potential readers of this book. The sequence in which I have treated various topics is closely connected with modern demands for strict mathematical rigour—demands which have become more and more acute over the years.

Historically speaking, therefore, the development of mathematical analysis has not been followed as closely as it might have been.

Thus, Chapter 1 is devoted to "real numbers", Chapter 3 to the "theory of limits", and it is not until Chapter 5 that I have commenced to give a systematic account of the differential and integral calculus. The historical sequence of events was, of course, the complete reverse. The differential and integral calculus were founded in the seventeenth century and developed in the eighteenth century, being applied to numerous important problems; the theory

of limits became the foundation-stone of mathematical analysis at the beginning of the nineteenth century and only in the second half of the nineteenth century did a clearly defined concept of real numbers come into being, which justified the most refined propositions of the theory of limits.

This book summarizes many years of experience in lecturing on mathematical analysis in Leningrad University.

G. M. FIKHTENGOL'TS

CHAPTER 1

REAL NUMBERS

§ 1. The set of real numbers and its ordering

1. Introductory remarks. The reader is familiar, from school courses of mathematics, with the rational numbers and their properties. However, already the demands of elementary mathematics result in a need for the extension of this number domain. In fact, among the rational numbers there frequently do not exist the roots of positive integers, for instance $\sqrt{2}$, i.e. *there is no rational fraction p/q where p and q are positive integers, the square of which is equal to 2.*

To prove this assertion assume the converse: let there exist a fraction p/q such that $(p/q)^2 = 2$. We may regard this fraction as irreducible, i.e. p and q have no common factors. Since $p = 2q^2$, p is an even number, $p = 2r$ (r is an integer) and, consequently, q is odd. Substituting for p its expression we find that $q^2 = 2r^2$ which implies that q is an even number. This contradiction proves our assertion.

Moreover, if we remain in the domain of rational numbers only, it is clear that in geometry not all segments may be provided with lengths. In fact, consider a square with side equal to the unit of length. *Its diagonal cannot have a rational length p/q*, since if this were the case, according to the Pythagoras theorem the square of its length would be 2, which we know to be impossible.

In the present chapter we intend to extend the domain of rational numbers by connecting with them numbers of a new kind—the irrational numbers.

The irrational numbers appear in mathematics—in the form of expressions containing roots—in medieval papers, but they were not regarded as genuine numbers. In the seventeenth century the coordinate method created by Descartes†

† René Descartes (1596–1650)—a celebrated French philosopher and scientist.

again raised the problem of the numerical description of geometric quantities. This induced a gradual growth of the concept of the common nature of irrational and rational numbers; it was finally formulated in the definition of a (positive) number given by Newton† in his *General Arithmetic* (1707): "By a number we understand not so much the set of unities as an abstract ratio of a quantity to another quantity of the same kind assumed to be unity." The integers and fractions express numbers commensurable with unity while the irrational numbers express those incommensurable with unity.

The mathematical analysis created in the seventeenth century and extensively developed throughout the whole of the eighteenth century was for a long time satisfied with this definition although it was alien to arithmetic and kept in the background the most important property of the extended number domain—its continuity (see Sec. 5 below). The critical trend in mathematics which arose at the end of the eighteenth and the beginning of the nineteenth century advanced the demand for a precise definition of the fundamental concepts of analysis and an exact proof of its basic statements. This in turn soon made it necessary to construct a logically sound theory of irrational numbers on the basis of a purely arithmetical definition. In the seventies of the nineteenth century a number of such theories were developed, superficially different in form but essentially equivalent. All these theories define an irrational number by connecting it with some infinite set of rational numbers.

2. Definition of irrational number. We shall give the theory of irrational numbers in the form due to Dedekind‡. This theory is based on the concept of a cut in the domain of rational numbers We consider the division of the set of all rational numbers into two non-empty (i.e. containing at least one number) sets A, A'; in other words we assume that

(1) every rational number belongs to one and only one set A or A'.

We call such a division a cut if one more condition is satisfied, namely:

(2) every number a of the set A is smaller than every number a' of the set A'.

Set A is called the lower class, and set A' the upper class. The cut will be denoted by $A|A'$.

The definition implies that any rational number smaller than a number a of the lower class also belongs to this class. Similarly, any rational number greater than a number a' of the upper class belongs to the upper class.

† Isaac Newton (1642–1727)—great English physicist and mathematician.

‡ Richard Dedekind (1831–1916)—a German mathematician.

Examples. (1) Define A as the set of all rational numbers a satisfying the inequality $a<1$, while set A' contains all numbers a' such that $a' \geqslant 1$.

It can easily be verified that we have in fact obtained a cut. The number unity belongs to class A' and obviously it is the smallest number of this set. On the other hand there is no greatest number in class A, since for any number a from A we can always indicate a rational number a_1 located between a and unity and, consequently, greater than a and also belonging to class A.

(2) The lower class A contains all rational numbers a smaller or equal to unity, $a \leqslant 1$, while the upper class contains all rational numbers a' greater than unity, $a' > 1$.

This is also a cut, and now the upper class has no smallest number whereas the lower does have the greatest (namely—unity).

(3) Class A contains all positive rational numbers a for which $a^2 < 2$, the number zero and all negative rational numbers, while class A' contains all positive rational numbers a' such that $a'^2 > 2$.

It is easily seen that we again have a cut. Now class A has no greatest number and class A' no smallest number. Let us, for instance, prove the first assertion (the second can be proved in an analogous way). Let a be an arbitrary positive number of class A; hence $a^2 < 2$. We prove that we can select a positive integer n such that

$$\left(a + \frac{1}{n}\right)^2 < 2,$$

so that the number $a + (1/n)$ also belongs to class A.

This inequality is equivalent to the following two:

$$a^2 + \frac{2a}{n} + \frac{1}{n^2} < 2,$$

$$\frac{2a}{n} + \frac{1}{n^2} < 2 - a^2.$$

The last inequality is certainly satisfied if n is such that $(2a + 1)/n < 2 - a^2$ for which it is sufficient to take

$$n > \frac{2a+1}{2-a^2}.$$

Thus, regardless of the value of the positive number a of class A, in the same class A there is always a greater number; since, for the numbers $a \leqslant 0$ this assertion is obvious, no number of class A is the greatest a in A.

Clearly, *there cannot exist a cut such that there is simultaneously a greatest number a_0 in the lower class and a smallest number a_0' in the upper class.*. In fact, assume that such a cut does exist. Then we take an arbitrary rational number c which lies between a_0 and a_0', $a_0 < c < a_0'$. The number c cannot belong to class A, since then a_0 would not be the smallest number in this class; for an analogous reason c cannot belong to class A' and this contradicts the property *(1)* of the cut, the latter property being a part of the definition of this concept.

Thus, cuts can be of three kinds illustrated in turn by Examples *(1)*, *(2)* and *(3)*, either:

(1) in the lower class A there is no greatest number and the upper class A' contains a smallest number r, or

(2) there is a greatest number r in the lower class A while the upper class A' has no smallest number, or, finally

(3) neither the lower class has a greatest number, nor the upper class a smallest number.

It is said in the first two cases that the cut is made by the rational number r (which is the boundary number between classes A and A') or that this cut defines the rational number r. In Examples *(1)* and *(2)* the number r was unity. In the third case a boundary number does not exist and the cut does not define any rational number. We now introduce new elements—the irrational numbers, by stating that *every cut of the form* (3) *defines an irrational number* α. This number α replaces the lacking boundary number; it seems to be introduced between all numbers a of class A and all numbers a' of class A'. In Example *(3)* this newly created number is evidently $\sqrt{2}$.

Without introducing any unified notation† for irrational numbers we shall always connect the irrational number α with the cut $A|A'$ in the domain of rational numbers, which defines it.

† We mean finite notation; the reader will become acquainted with a kind of infinite notation in § 1.4. Irrational numbers are usually denoted by forms depending on their origin and role, e.g. $\sqrt{2}$, log 5, sin 10°, etc.

For consistency it will frequently be convenient to do the same for a rational number r. However, for every number r there exist two cuts defining it; in both cases the numbers $a < r$ are contained in the lower class, while the numbers $a' > r$ belong to the upper class, but the number r itself can be referred either to the lower class (then r is the greatest number there) or to the upper (in this class r is the smallest number). For definiteness we agree once for all, when speaking of the cut defining the rational number r, to introduce this number into the upper class.

The rational and irrational numbers are jointly called real numbers. The concept of a real number is one of the basic concepts of mathematical analysis, and indeed of the whole of mathematics.

3. Ordering of the set of real numbers. *Two irrational numbers α and β defined by the cuts $A|A'$ and $B|B'$, respectively, are said to be equal if and only if the two cuts are identical*; incidentally, it is sufficient to require the identity of the lower classes A and B, since then the upper classes A' and B' are automatically identical. This definition can be preserved for the case of rational α and β. In other words, *if two rational numbers α and β are equal, the cuts defining them are identical, and, conversely, the identity of the cuts implies the equality of the numbers α and β.* It is evident that in this case the above condition concerning rational numbers is to be taken into account.

We proceed to establish the concept of "greater than" with respect to real numbers. For rational numbers this concept is known from elementary mathematics. For a rational number r and an irrational number α the concept "greater than" was actually established in Sec. 2, namely, if α is defined by the cut $A|A'$ we regard α to be greater than all rational numbers belonging to the class A, all numbers of class A' being greater than α.

Consider now two irrational numbers α and β, α being defined by the cut $A|A'$, and β by $B|B'$. We regard as greater the number for which the lower class is greater. More precisely, *we say that $\alpha > \beta$ if class A wholly contains class B and does not coincide with it.* (It is evident that this condition is equivalent to stating that class B' wholly contains class A' and does not coincide with it.) It can easily be

verified that this definition can be preserved also for the cases when one of the numbers α, β (or even both) is rational.

The concept "smaller than" is now introduced as a dependent property. Thus, we say that $\alpha < \beta$ if and only if $\beta > \alpha$.

Our definitions imply that:

For any pair of real numbers α and β, one and only one of the relations

$$\alpha = \beta, \quad \alpha > \beta, \quad \alpha < \beta$$

holds.

Furthermore,

$$\alpha > \beta, \ \beta > \gamma \quad \textit{imply that} \quad \alpha > \gamma.$$

It is also obvious that

$$\alpha < \beta, \ \beta < \gamma \quad \textit{imply that} \quad \alpha < \gamma.$$

Let us finally establish two auxiliary assertions which will frequently be useful later.

LEMMA 1. *For any pair of real numbers α and β, where $\alpha > \beta$, there can always be found a real, and even in particular a rational, number r which lies between them, i.e. $\alpha > r > \beta$ (and, consequently, an infinite set of such rational numbers).*

Since $\alpha > \beta$ the lower class A of the cut defining the number α wholly contains the lower class B for the number β and it is not identical with B. Hence a rational number r can be found in A which does not belong to B and, consequently, belongs to B'; for this number we have

$$\alpha > r \geqslant \beta$$

(equality could occur only if β were rational). But, since there is no greatest number in A, the equality can be eliminated, increasing r if necessary.

LEMMA 2. *Consider two real numbers α and β. If, for an arbitrary rational number $e > 0$, the numbers α and β can be contained within the same rational bounds*

$$s' \geqslant \alpha \geqslant s, \quad s' \geqslant \beta \geqslant s,$$

the difference of which is smaller than e, i.e.

$$s' - s < e,$$

then the numbers α and β are necessarily equal.

The proof is carried out by assuming the converse. Suppose for instance that $\alpha > \beta$. According to Lemma 1 we insert between α and β two rational numbers r and $r' > r$ such that

$$\alpha > r' > r > \beta.$$

Then for two arbitrary numbers s and s' between which lie α and β the following inequalities are obviously valid:

$$s' > r' > r > s, \quad \text{so} \quad s' - s > r' - r > 0,$$

and hence the difference $s' - s$ cannot be, for instance, smaller than the number $e = r' - r$, which contradicts the assumption of the lemma. This proves the theorem.

4. Representation of a real number by an infinite decimal fraction. We seek a representation, the fractional part of which is positive, while the entire part may be positive, negative or zero.

We first assume that the real number α to be considered is neither an integer nor a finite decimal fraction. We seek its decimal approximation. If the number is defined by the section $A|A'$ then it is easy to show that in class A a number M can be found which is an integer, and in class A' an integer $N > M$. Adding unity repeatedly to M, we must eventually arrive at two consecutive integers C and $C+1$ such that

$$C < \alpha < C + 1.$$

The number C can be positive, negative or zero.

Further, if we divide the interval between C and $C+1$ into ten equal parts by the numbers

$$C.1;\ C.2;\ \ldots;\ C.9,$$

then α belongs to one (and only one) of the partial intervals, and we arrive at two numbers differing by $1/10$: $C.c_1$ and $C.c_1 + (1/10)$ for which

$$C.c_1 < \alpha < C.c_1 + \frac{1}{10}.$$

Continuing this procedure, after having determined $n-1$ digits $c_1, c_2, \ldots, c_{n-1}$ the nth digit c_n is defined by the inequalities

$$C.c_1 c_2 \ldots c_n < \alpha < C.c_1 c_2 \ldots c_n + \frac{1}{10^n}. \tag{1}$$

Thus, in the course of finding the decimal approximation of the number α we have constructed an integer C and an infinite sequence of digits $c_1, c_2, \ldots, c_n, \ldots$. *The infinite decimal fraction constructed from them, i.e. the symbol*

$$C.c_1c_2 \ldots c_n \ldots, \tag{2}$$

may be regarded as a representation of the real number α.

In the excluded case where α itself is an integer or in general a finite decimal fraction, we can in a similar way find successively the number C and the digits $c_1, c_2, \ldots, c_n, \ldots$, but on the basis of relations

$$C.c_1c_2 \ldots c_n \leqslant \alpha \leqslant C.c_1c_2 \ldots c_n + \frac{1}{10^n}, \tag{1a}$$

which are more general than (1). This is due to the fact that, at some instant, the number α coincides with one of the ends of the interval in which it lies; it will be arbitrarily the left or the right end. From then on, the equality (1a) occurs on the left or on the right, respectively. Thus, the following digits are all zeros or nines, depending on which contingency arises. In this case the number α has a double representation: one with recurring zero and the second with recurring nine, for instance

$$2.718 = 2.718000 \ldots = 2.717999 \ldots,$$

$$-2.718 = \bar{3}.282 = \bar{3}.282000 \ldots = \bar{3}.281999 \ldots.$$

The difference between the decimal approximations

$$C.c_1c_2 \ldots c_n \quad \text{and} \quad C.c_1c_2 \ldots c_n + \frac{1}{10^n},$$

with excess and defect respectively, is equal to $1/10^n$, and as n increases this can be made smaller than any rational number $e > 0$. In fact, since there is a finite number of positive integers not exceeding the number $1/e$, the inequality $10^n \leqslant 1/e$, or equivalently, $1/10^n \geqslant e$, can be satisfied for only a finite number of values of n; for all other values we have

$$\frac{1}{10^n} < e.$$

In view of this and Lemma 2 it is seen that the number β, not equal to α, cannot satisfy the same set of inequalities (1) and (1a) as α,

and consequently it has a representation in the form of an infinite decimal fraction distinct from that of the number α.

In particular, this implies that the representation of a number not equal to any finite decimal fraction has neither recurring zero nor recurring nine, for any fraction with recurring zero or recurring nine expresses a finite decimal fraction explicitly.

It can be proved that if we take arbitrarily the infinite fraction (2), then there exists a real number α for which the fraction (2) is the exact representation. Evidently, it is sufficient to construct the number α in such a way that all inequalities (1a) are satisfied. Hence, introducing for brevity the notation

$$C_n = C.c_1c_2 \dots c_n \quad \text{and} \quad C_n' = C.c_1c_2 \dots c_n + \frac{1}{10^n},$$

we observe that every fraction C_n is smaller than every fraction C_m' (not only for $m = n$ but also for $m \gtrless n$). Now making the cut in the domain of rational numbers we place in the upper class A' all rational numbers a' which are greater than all C_n (for instance all numbers C_m'), and in the lower class A all the remaining numbers (for instance the numbers C_n themselves). It can easily be verified that this is in fact a cut; it defines the required real number α.

In fact, since α is the boundary number between two classes, in particular

$$C_n \leqslant \alpha \leqslant C_n'.$$

Now the reader can regard the real numbers as infinite decimal fractions. It is known from school courses that a recurring infinite decimal fraction represents a rational number, and, conversely, every rational number can be expanded into a recurring decimal fraction. Thus, the representations of the newly introduced irrational numbers are non-recurring infinite fractions. This representation can also be a starting point for constructing a theory of irrational numbers.

Remark. Subsequently we shall have to make use of approximate rational values a and a' of the real number α

$$a < \alpha < a',$$

the difference between which is smaller than an arbitrarily small number $e > 0$. For a rational α the existence of the numbers a and

a' is obvious; for an irrational α for a and a' we could take for instance the decimal approximations C_n and C_n' for a sufficiently large n.

5. Continuity of the set of real numbers. We now proceed to consider a very important property of the set of all real numbers; it is this property that makes it essentially different from the set of rational numbers. Investigating cuts in the set of rational numbers we found that sometimes there was no explicit boundary number in this set which could be said to define the cut. It is exactly this incompleteness of the set of rational numbers (i.e. the presence of gaps in it) which constitutes the basis of introducing new numbers—the irrational numbers. We shall now examine cuts in the set of all real numbers. By such a cut we understand the division of the set into two non-empty sets A, A' where:

(1) every real number belongs to one and only one of the sets A, A' and moreover,

(2) every number α *of the set* A *is smaller than any number* α' *of the set* A'.

There arises the question—does there always exist for such a cut in the set of real numbers a boundary number giving rise to this cut, or do there exist gaps in the considered set as well (which could force us to introduce still new numbers)?

It turns out that in fact there are no such gaps.

THE FUNDAMENTAL THEOREM (Dedekind's theorem). *For any cut* $A|A'$ *in the set of real numbers there exists a real number* β *which gives rise to this cut. This number* β *is either* $\{1\}$ *the greatest in the lower class* A, *or* $\{2\}$ *the smallest in the upper class* A'.

This property of the set of real numbers is called its completeness or continuity.

Proof. Denote by A the set of all rational numbers belonging to A and by A' the set of all rational numbers belonging to A'. It can easily be found that the sets A and A' give rise to a cut in the set of all rational numbers.

This cut $A|A'$ defines a real number β. It should belong to one of the classes A, A'; assume for instance that β belongs to the lower class A and let us prove that then case $\{1\}$ occurs, namely β is the greatest number in the class A. In fact, were this not the case, there

would exist another number α_0 of this class greater than β. Introduce (on the basis of Lemma 1) a rational number r between α_0 and β such that

$$\alpha_0 > r > \beta.$$

r belongs to class A and, consequently, also to class A. We have arrived at a contradiction—the rational number r belonging to the lower class of the cut defining the number β is greater than this number! This proves our assertion.

Similar reasoning shows that if β belongs to the upper class A', then case $\{2\}$ occurs.

Remark. Simultaneous existence in class A of the greatest number and in class A' of the smallest is impossible; this fact can be established as for cuts in the domain of rational numbers (with the help of Lemma 1).

6. Bounds of number sets. We now make use of the fundamental theorem [Sec. 5] in order to establish some concepts which play important roles in modern analysis. They will immediately be useful in considering arithmetic operations on real numbers.

Imagine an arbitrary infinite set† of real numbers; it can be prescribed in an arbitrary way. Such sets are, for instance, the set of positive integers, the set of all proper fractions, the set of all real numbers between the numbers 0 and 1, the set of roots of the equation $\sin x = 1/2$, etc.

We denote any of the numbers of the set by x; thus x is a typical symbol for the numbers of the set; the set of the numbers x itself is denoted by $\mathcal{X} = \{x\}$.

If, for the considered set $\{x\}$, there exists a number M such that all $x \leqslant M$, it is said that the set is bounded above (by the number M); the number M is itself called an upper bound of the set $\{x\}$. For instance, the set of proper fractions is bounded above by unity or by any number greater than unity; the sequence of positive integers is not bounded above.

Similarly, we have: if a number m can be found, such that all $x \geqslant m$, then it is said that the set $\{x\}$ is bounded below (by the

† All that is said below is also valid for finite sets, but this case is of no interest.

number m), and the number m is itself called a lower bound of the set $\{x\}$. For instance, the sequence of positive integers is bounded below by the number 1 or by any number smaller than 1; the set of proper fractions is bounded below by the number 0 or by any negative number.

A set bounded above (below) can at the same time be bounded below (above). Thus, the set of proper fractions is bounded above and below, while the sequence of positive integers is bounded below but not bounded above.

If a set is not bounded above (below), its upper (lower) bound is said to be the "improper number" $+\infty$ $(-\infty)$. The symbols $+\infty$ and $-\infty$ read: "plus infinity" and "minus infinity". For these "improper" or "infinite" numbers we assume that

$$-\infty < +\infty \quad \text{and} \quad -\infty < \alpha < +\infty,$$

regardless of the value of the real ("finite") number α.

If a set is bounded above, i.e. it has a finite upper M, then it also has an infinite set of upper bounds (for instance any, number greater than M is evidently also an upper bound). From all upper bounds the most important is the smallest, which is called the *least upper bound.* Similarly, if the set is bounded from below, then the greatest of all lower bounds is called the *greatest lower bound.* Thus, for the set of all proper fractions the greatest lower bound and the least upper bound are the numbers 0 and 1, respectively.

The following problem arises: for a set bounded above (below) does there always exist a least upper (greatest lower) bound? In fact, since in this case there is an infinite number of upper (lower) bounds and among the infinite set of numbers there is not always a smallest (greatest)†, the very existence of such a smallest (greatest) number among all upper (lower) bounds of the set under consideration requires a proof.

THEOREM. *If the set* $\mathfrak{X} = \{x\}$ *is bounded above (below), then it has also a least upper (greatest lower) bound*‡.

† There is none, for instance, among all proper fractions.

‡ This theorem, in a different formulation, was first announced in 1817 by a Czech philosopher and mathematician, Bernhardt Bolzano (1781–1848). A rigorous proof became possible only after making more precise the concept of real number.

Proof. We carry out the proof for the upper bound. Consider two cases:

(1) We first assume that among the numbers x of the set $\mathcal{X}$ there is a greatest $\bar{x}$. Then all numbers of the set satisfy the inequality $x \leqslant \bar{x}$, i.e. $\bar{x}$ is the upper bound for $\mathcal{X}$. On the other hand $\bar{x}$ belongs to $\mathcal{X}$; consequently, for any upper bound M the inequality $\bar{x} \leqslant M$ holds. Hence we infer that $\bar{x}$ is the least upper bound of the set $\mathcal{X}$.

(2) Assume now that among the numbers x of the set $\mathcal{X}$ there is no greatest number. We construct the cut in the domain of all real numbers in the following way. To the upper class A' we refer all upper bounds α' of the set $\mathcal{X}$ and to the lower class A all remaining real numbers α. Then all numbers x of the set $\mathcal{X}$ belong to class A, since, according to the assumption, none of them is the greatest. Thus, both classes A and A' are non-empty. This division is in fact a cut, since all real numbers are distributed over classes and every number of class A' is greater than any number of class A. According to Dedekind's theorem [Sec. 5] there should exist a real number β giving rise to the cut. All numbers x belonging to class A do not exceed this "boundary" number β, i.e. β is an upper bound for all x and, consequently, it belongs itself to class A' and is the smallest in this class. Thus β is the smallest of all upper bounds, and is thus the required least upper bound of the set $\mathcal{X} = \{x\}$.

In an entirely similar way we prove the second half of the theorem (concerning the existence of a greatest lower bound).

If M^* is the least upper bound of a number set $\mathcal{X} = \{x\}$ then for all x we have that

$$x \leqslant M^*.$$

We now take an arbitrary number α smaller than M^*. Since M^* is the smallest of the upper bounds, the number α certainly is not an upper bound for the set $\mathcal{X}$, i.e. there is a number x' from $\mathcal{X}$ such that

$$x' > \alpha.$$

These two inequalities fully describe the least upper bound M^* of the set $\mathcal{X}$.

In a similar way the greatest lower bound m^* of the set $\mathcal{X}$ is described by the fact that for all x

$$x \geqslant m^*,$$

and, for any number β greater than m^*, a number x'' can be found from $\mathcal{X}$, such that

$$x'' < \beta.$$

We denote the exact upper bound M^* and the exact lower bound m^* of the number set $\mathcal{X}$ by the following symbols:

$$M^* = \sup \mathcal{X} = \sup \{x\}, \quad m^* = \inf \mathcal{X} = \inf \{x\}$$

(from the Latin: *supremum*—greatest, *infimum*—smallest).

We note an obvious conclusion which will frequently be used below:

If all the numbers x of a set satisfy the inequality $x \leqslant M$, then $\sup\{x\} \leqslant M$.

In fact, the number M is one of the upper bounds of the set and, hence, the smallest of all upper bounds does not exceed M.

Similarly, *the inequality $x \geqslant m$ implies that* $\inf \{x\} \geqslant m$.

Finally, let us agree that if the set $\mathcal{X} = \{x\}$ is not bounded above, we say that its least upper bound is $+\infty$: $\sup\{x\} = +\infty$. Similarly, if the set $\mathcal{X} = \{x\}$ is not bounded below it is said that its greatest lower bound is $-\infty$: $\inf\{x\} = -\infty$.

§ 2. Arithmetical operations over real numbers

7. Definition and properties of a sum of real numbers. Consider two real numbers α and β. We shall examine the rational numbers a, a' and b, b' which satisfy the inequalities

$$a < \alpha < a', \quad b < \beta < b'. \tag{1}$$

By the sum of the numbers α and β we understand a real number γ which lies between all sums of the form $a+b$, and all sums of the form $a'+b'$, i.e.

$$a + b < \gamma < a' + b'. \tag{2}$$

Let us first establish that such a number γ exists for an arbitrary pair of real numbers α, β.

Consider the set of all possible sums $a+b$. This set is bounded above for instance, by an arbitrary sum of the form $a'+b'$. Set [Sec. 6]

$$\gamma = \sup \{a + b\}.$$

Then $a+b \leqslant \gamma$ and at the same time $\gamma \leqslant a'+b'$.

For any rational numbers a, b, a', b' satisfying (1) the numbers a, b can always be increased and the numbers a', b' decreased, preserving the above conditions; thus, in the above inequalities with $<$ replaced by $\leqslant$, in no case can equality occur. Thus, the number γ satisfies the definition of the sum.

However, the following problem arises: is the sum $\gamma = \alpha+\beta$ uniquely defined by the inequalities (2)? To establish the uniqueness of the sum let us select (see Sec. 4, Remark) rational numbers a, a', b, b' in such a way that

$$a'-a<e \quad \text{and} \quad b'-b<e,$$

where e is an arbitrarily small rational positive number. Hence

$$(a'+b')-(a+b) = (a'-a)+(b'-b)<2e,$$

i.e. this difference can be made arbitrarily small†. Then, however, by Lemma 2, there exists only one number between the sums $a+b$ and $a'+b'$.

Finally, observe that if the numbers α and β are both rational, then their ordinary sum $\gamma = \alpha+\beta$ obviously satisfies inequalities (2). Thus, the above general definition of the sum of two real numbers does not contradict the previous definition of the sum of two rational numbers.

For real numbers all basic properties of addition are valid, namely

$$(1)\ \alpha+\beta = \beta+\alpha, \quad (2)\ (\alpha+\beta)+\gamma = \alpha+(\beta+\gamma), \quad (3)\ \alpha+0 = \alpha$$

and, finally

$$(4)\ \alpha>\beta \quad \textit{implies that} \quad \alpha+\gamma>\beta+\gamma.$$

These can easily be proved from the definition of the sum given above and with the help of well-known properties of rational numbers; we shall not dwell on this problem. The last property justifies the term-by-term addition of two inequalities.

8. Symmetric numbers. Absolute quantity. We now prove that *for any real number α there exists (symmetric to it) a number $-\alpha$ such that* $\alpha+(-\alpha) = 0$.

It is sufficient to consider the case of an irrational number α.

Assuming that the number α is defined by the cut $A|A'$ we define the number $-\alpha$ as follows. In the lower class A of the number $-\alpha$ we place all rational numbers $-a'$ where a' is an arbitrary number of class A', and in the upper class A' of this number all numbers $-a$ where a is an arbitrary number of class A. It is readily observed that the constructed division is a cut and in fact it defines a real (in our case irrational) number which we denote by $-\alpha$.

We now establish that this number satisfies the required condition. Using the definition of the number $-\alpha$ itself we observe that the sum $\alpha+(-\alpha)$ is a real number lying between the numbers of the form $a-a'$ and $a'-a$ where a and a' are rational and $a<\alpha<a'$. But, obviously,

$$a-a'<0<a'-a,$$

† The number $2e$ becomes smaller than any number $e'>0$, provided we take $e<e'/2$.

whence the number 0 also lies between the above numbers. In view of the uniqueness of the number possessing this property we have

$$\alpha + (-\alpha) = 0,$$

which completes the proof.

Notice that a number symmetric to a given number is unique and has the properties

$$-(-\alpha) = \alpha, \quad -(\alpha+\beta) = (-\alpha)+(-\beta).$$

By means of the concept of a symmetric number we can introduce the idea of the subtraction of real numbers as an operation inverse to addition. We call the difference between α and β (we denote it by $\alpha-\beta$) the number γ which satisfies the condition

$$\gamma+\beta=\alpha \quad \text{or} \quad (\beta+\gamma=\alpha).$$

On the basis of the properties of addition it can easily be proved that such a number is $\gamma=\alpha+(-\beta)$; in fact,

$$\begin{aligned}\gamma+\beta &= [\alpha+(-\beta)]+\beta = \alpha+[(-\beta)+\beta] \\ &= \alpha+[\beta+(-\beta)] = \alpha+0=\alpha.\end{aligned}$$

This establishes also the uniqueness of the difference.

The property *(4)* of Sec. 7 now makes it possible to make a useful remark about the equivalence of the inequalities

$$\alpha>\beta \quad \text{and} \quad \alpha-\beta>0,$$

and this enables us to establish that the inequality $\alpha>\beta$ implies the inequality $-\alpha<-\beta$.

Finally, the concept of a symmetric number is connected with the concept of the absolute value of a number. The very construction of the symmetric number implies that for $\alpha>0$ we necessarily have $-\alpha<0$ and that $\alpha<0$ implies $-\alpha>0$. In other words only if $\alpha\neq 0$, one (and only one) of the numbers α and $-\alpha$ is greater than zero; this number is called the absolute value of the number α and the number $-\alpha$; it is denoted by the symbol

$$|\alpha| = |-\alpha|.$$

The absolute value of the number zero is assumed to be equal to zero, i.e. $|0|=0$.

For the sake of future considerations we now make two more remarks concerning absolute values.

First we establish that the inequality $|\alpha|<\beta$ (where evidently $\beta>0$) is equivalent to the double inequality $-\beta<\alpha<\beta$.

In fact, it follows from $|\alpha|<\beta$ that, at the same time, $\alpha<\beta$ and $-\alpha<\beta$, i.e. $\alpha>-\beta$. Conversely, if it is known that $\alpha<\beta$ and $\alpha>-\beta$, then we have simultaneously $\alpha<\beta$ and $-\alpha<\beta$; but one of the numbers α, $-\alpha$ is $|\alpha|$ so that $|\alpha|<\beta$.

Similarly, the following inequalities are equivalent:

$$|\alpha|\leqslant\beta \quad \text{and} \quad -\beta\leqslant\alpha\leqslant\beta.$$

We now prove the useful inequality:

$$|\alpha+\beta|\leqslant|\alpha|+|\beta|.$$

Adding term-by-term the obvious inequalities

$$-|\alpha|\leqslant\alpha\leqslant|\alpha| \quad \text{and} \quad -|\beta|\leqslant\beta\leqslant|\beta|,$$

we obtain

$$-(|\alpha|+|\beta|)\leqslant\alpha+\beta\leqslant|\alpha|+|\beta|,$$

whence, in view of the remark made above, the required inequality follows.

By means of mathematical induction the inequality proved above can be extended to an arbitrary number of terms. Moreover, it implies that

$$|\alpha+\beta|\geqslant|\alpha|-|\beta|,$$

and also that

$$|\alpha|-|\beta|\leqslant|\alpha-\beta|\leqslant|\alpha|+|\beta|.$$

Since at the same time

$$|\beta|-|\alpha|\leqslant|\alpha-\beta|,$$

we obviously have

$$\big||\alpha|-|\beta|\big|\leqslant|\alpha-\beta|.$$

All these inequalities will frequently be of use later.

9. Definition and properties of a product of real numbers. We now consider the multiplication of real numbers, first confining ourselves to positive numbers. Consider two such numbers α and β. We shall here also examine all rational numbers satisfying inequalities (1) where these numbers are taken to be positive.

By the product $\alpha\beta$ of two positive real numbers α and β we mean a real number γ which lies between all products ab and all products $a'b'$:

$$ab<\gamma<a'b'. \tag{3}$$

To prove the existence of such a number γ we take the set of all products ab; it is bounded above by any of the products $a'b'$. If we set

$$\gamma=\sup\{ab\},$$

then, of course, $ab\leqslant\gamma$ but at the same time $\gamma\leqslant a'b'$.

The possibility of increasing the numbers a, b and decreasing a', b' (as in the case of the sum) makes it possible to exclude the equality sign, and hence the number γ satisfies the definition of the product.

The uniqueness of the product can be proved as follows; let us select the rational numbers a, a' and b, b', such that (see Sec. 4, Remark)

$$a'-a<e \quad \text{and} \quad b'-b<e,$$

where e is an arbitrary small positive rational number. We can here assume that the numbers a and b are positive and the numbers a' and b' do not exceed some previously fixed numbers a_0' and b_0', respectively. Then the difference

$$a'b'-ab=a'(b'-b)+b(a'-a)<(a_0'+b_0')e,$$

i.e. it can also be made arbitrarily small† and thus, by Lemma 2, we see that the inequalities (3) can be satisfied by only one number, γ.

† Observe that $(a_0'+b_0')e$ becomes smaller than any number $e'>0$, provided we take $e<e'/(a_0'+b_0')$.

If both positive numbers α and β are rational, their ordinary product evidently satisfies the inequalities (3), i.e. it is in accord with the general definition of two real numbers; thus, there is no contradiction.

Finally, in order to define the product of an arbitrary pair of real numbers (not necessarily positive) we shall require the following results.

First we set

$$\alpha \cdot 0 = 0 \cdot \alpha = 0$$

regardless of the number α.

If now both factors are different from zero we take for the ordinary "rule of signs":

$$\alpha \cdot \beta = |\alpha| \cdot |\beta| \quad \text{if} \quad \alpha, \beta \text{ are of the same sign,}$$
$$\alpha \cdot \beta = -(|\alpha| \cdot |\beta|) \quad \text{if} \quad \alpha, \beta \text{ are of different signs}$$

(this is in agreement with the product of two positive numbers $|\alpha|$ and $|\beta|$, which we already know).

As in the case of rational numbers, for arbitrary real numbers the following properties hold:

$$(1)\ \alpha \cdot \beta = \beta \cdot \alpha, \quad (2)\ (\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma), \quad (3)\ \alpha \cdot 1 = \alpha,$$
$$(4)\ (\alpha + \beta) \cdot \gamma = \alpha \cdot \gamma + \beta \cdot \gamma,$$

and also

(5) from $\alpha > \beta$ and $\gamma > 0$ it follows that $\alpha \cdot \gamma > \beta \cdot \gamma$.

By means of the last property the term-by-term multiplication of two inequalities with positive terms is justified.

If we define the quotient α/β of the numbers α and β as the number γ which satisfies the property

$$\gamma \cdot \beta = \alpha \quad (\text{or } \beta \cdot \gamma = \alpha),$$

we can establish the existence and uniqueness of the quotient, provided only that the divisor β is different from zero.

To end this survey of arithmetical operations over real numbers let us emphasize once more that all the fundamental properties of rational numbers which constitute the basis of elementary algebra hold also for real numbers. Consequently, for real numbers, all those rules of algebra which concern the arithmetical operations and combination of equalities and inequalities are valid.

§ 3. Further properties and applications of real numbers

10. Existence of a root. Power with a rational exponent. The definition of multiplication (and division) of real numbers leads directly to the definition of a power with an integral positive (and negative) exponent. Proceeding to the power with any rational exponent we first consider the problem of the existence of a root.

We remember that the absence in the domain of rational numbers of the simplest roots was one of the reasons for extension of this domain; let us

see to what extent the above extension has filled the old gaps (without creating new ones).

Let α be an arbitrary real number and n a positive integer.

It is known that by the root of nth degree of the number α we mean a real number ξ such that

$$\xi^n = \alpha.$$

We confine ourselves to positive α and we seek a positive number ξ which satisfies this relation, i.e. the so-called *arithmetical value* of the root. We shall prove that such a number ξ' always exists and that it is unique.

Incidentally, the last statement concerning the uniqueness of the number ξ follows at once from the fact that to distinct positive numbers there correspond distinct powers: if $0 < \xi < \xi'$, then $\xi^n < \xi'^n$.

If there exists a positive rational number r, the nth degree of which is equal to α, then it is the required number ξ. Hence, it is hereafter sufficient to confine ourselves to the assumption that there is no such rational number.

Let us now construct the cut $X|X'$ in the domain of all rational numbers in the following way. To the class X we refer all negative rational numbers zero, and also those positive rational numbers x for which $x^n < \alpha$. The class X' contains the positive rational numbers x' for which $x'^n > \alpha$.

It is readily seen that these classes are non-empty and that X contains positive numbers. If we take for instance the positive integer m such that $1/m < \alpha < m$, then certainly we also have $1/m^n < \alpha < m^n$ and hence the number $1/m$ belongs to X and the number m to X'.

Other requirements for a cut can be verified directly.

Now let ξ be a number defined by the cut $X|X'$; we prove that $\xi^n = \alpha$, i.e. that $\xi = \sqrt[n]{\alpha}$.

Regarding ξ^n as the product of n factors equal to ξ we infer, on the basis of the definition of the product of positive real numbers, that if x and x' are rational numbers for which

$$0 < x < \xi < x',$$

then

$$x^n < \xi^n < x'^n.$$

Since it is evident that x belongs to class X and x' to class X', from the definition of these classes we also have that

$$x^n < \alpha < x'^n.$$

But the difference $x' - x$ can be made smaller than any number $e > 0$ (see Sec. 4, Remark) and thus we can take x' smaller than any previously fixed number x'_0. Thus the difference

$$x'^n - x^n = (x' - x)(x'^{n-1} + x \cdot x'^{n-2} + \ldots + x^{n-1}) < e \cdot n x_0'^{n-1},$$

i.e. it also can be made arbitrarily small†. Hence, according to Lemma 2 we obtain the equality of the numbers ξ^n and α.

---

† Observe that the number $e \cdot n x_0'^{n-1}$ becomes smaller than any number $e' > 0$, provided we take $e < e'/(n x_0'^{n-1})$.

After having proved the existence of the root we establish, by the ordinary method, the concept of the power with an arbitrary rational exponent r and it can be verified that for such powers the ordinary rules derived in elementary algebra are valid:

$$\alpha^r \cdot \alpha^{r'} = \alpha^{r+r'}, \quad \alpha^r : \alpha^{r'} = \alpha^{r-r'},$$

$$(\alpha^r)^{r'} = \alpha^{r \cdot r'}, \quad (\alpha\beta)^r = \alpha^r \cdot \beta^r,$$

$$\left(\frac{\alpha}{\beta}\right)^r = \frac{\alpha^r}{\beta^r}, \text{ etc.}$$

Let us emphasize that for $\alpha > 1$ the power α^r increases with increasing rational exponent r.

11. Power with an arbitrary real exponent. Consider now the definition of a *power of an arbitrary real (positive) number* α *with an arbitrary real exponent* β.

Introduce the powers of the number α

$$\alpha^b \quad \text{and} \quad \alpha^{b'}$$

with rational exponents b and b' which satisfy the inequalities

$$b < \beta < b'.$$

We define the power of number $\alpha > 1$† *with exponent* β *(and denote it by* α^β*) as the real number* γ *lying between the powers* α^b *and* $a^{b'}$.

$$\alpha^b < \gamma < \alpha^{b'}. \tag{1}$$

It can easily be verified that such a number always exists. In fact, the set of powers $\{\alpha^b\}$ is bounded above, for instance by any power $\alpha^{b'}$. We then take [Sec. 6]:

$$\gamma = \sup_{b < \beta} \{\alpha^b\}.$$

For this number

$$\alpha^b \leqslant \gamma \leqslant \alpha^{b'}.$$

In fact, however, the equality sign is superfluous, in view of the possibility of increasing b and decreasing b', so that the number γ satisfies conditions (1)

We now prove that the number defined by these conditions is unique.

We first note that Lemma 2 of Sec. 3 also holds in this case, if we abandon the requirement that the numbers s, s' and e necessarily be rational; the proof remains the same.

Then we establish one simple but frequently useful inequality: if n is a positive integer greater than unity and $\gamma > 1$ then

$$\gamma^n > 1 + n(\gamma - 1). \tag{2}$$

---

† We may confine ourselves to this case; for $\alpha < 1$ we write for instance $\alpha^\beta = (1/\alpha)^{-\beta}$.

In fact, setting $\gamma = 1 + \lambda$ where $\lambda > 0$, by Newton's binomial formula we obtain

$$(1 + \lambda)^n = 1 + n\lambda + \ldots;$$

since the unwritten terms are positive we have

$$(1 + \lambda)^n > 1 + n\lambda,$$

which is equivalent to inequality (2).

Putting in this inequality $\gamma = \alpha^{1/n}$ $(\alpha > 1)$ we obtain the inequality

$$\alpha^{\frac{1}{n}} - 1 < \frac{\alpha - 1}{n}, \tag{3}$$

which we shall now use.

We know [Sec. 4, Remark] that the numbers b and b' can be chosen so that the difference $b' - b$ is smaller than $1/n$ for a previously chosen arbitrary positive integer n; then, by inequality (3):

$$\alpha^{b'} - \alpha^b = \alpha^b(\alpha^{b'-b} - 1) < \alpha^b(\alpha^{\frac{1}{n}} - 1) < \alpha^b \frac{\alpha - 1}{n}.$$

Finally, denoting by b'_0 any of the numbers b',

$$\alpha^{b'} - \alpha^b < \alpha^{b'_0} \frac{\alpha - 1}{n}.$$

By altering n this expression can be made smaller than an arbitrarily small positive number ε; for this it is sufficient to take

$$n > \frac{\alpha^{b'_0}(\alpha - 1)}{\varepsilon}.$$

In this case, according to the generalized Lemma 2, no two distinct numbers γ can lie between the bounds α^b and $\alpha^{b'}$.

If β is rational, the above definition leads back to the original meaning of the symbol α^β.

It can easily be verified that for a power with an arbitrary real exponent all ordinary rules for a power hold, and also that for $\alpha > 1$ the power α^β increases when the real exponent β increases.

12. Logarithms. Making use of the definition of the power with an arbitrary real exponent, we can now easily establish the existence of a logarithm for an arbitrary positive real number γ for a positive base α not equal to unity (we assume for instance that $\alpha > 1$).

If there exists a rational number r such that

$$\alpha^r = \gamma,$$

then r is the required logarithm. Assume therefore that such a rational number r does not exist.

Then we can form the cut $B|B'$ in the domain of all rational numbers by the following method. In the class B we place the rational numbers b for which $\alpha^b < \gamma$ and in class B' the rational numbers b' for which $\alpha^{b'} > \gamma$.

We prove that the classes B and B' are non-empty. In view of inequality (2)

$$\alpha^n > 1 + n(\alpha - 1) > n(\alpha - 1),$$

and it is sufficient to take

$$n > \frac{\gamma}{\alpha - 1},$$

in order that $\alpha^n > \gamma$; such a positive integer n belongs to class B'. At the same time we have

$$\alpha^{-n} = \frac{1}{\alpha^n} < \frac{1}{n(\alpha - 1)}$$

and it is sufficient to take

$$n > \frac{1}{\gamma(\alpha - 1)},$$

to have $\alpha^{-n} < \gamma$ and to ensure that the number n belongs to class B.

The remaining requirements for a cut are also satisfied.

The constructed cut $B|B'$ defines a real number β which is the "boundary" number between the numbers of the two classes. According to the definition of the power we have:

$$\alpha^b < \alpha^\beta < \alpha^{b'} \quad (b < \beta < b'),$$

and α^β is the *only* number which satisfies all such inequalities. But for the number γ we have (according to the very definition of the cut):

$$\alpha^b < \gamma < \alpha^{b'}.$$

Consequently,

$$\alpha^\beta = \gamma \quad \text{and} \quad \beta = \log_\alpha \gamma,$$

and the existence of the logarithm is proved.

13. Measuring segments. The impossibility of providing all segments with lengths was also one of the most important reasons for introducing irrational numbers. We now prove that the above extension of the concept of numbers is sufficient to solve the problem of measuring segments.

First we formulate the problem itself.

It is required to associate with every rectilinecr segment A a positive real number $l(A)$ which will be called "the length of segment A", in such a way that:

(1) a prescribed segment E (the standard of length) has length equal to unity, $l(E) = 1$;

(2) equal segments have the same lengths;

(3) in the addition of segments the length of the sum is equal to the sum of the lengths of the added segments:

$$l(A + B) = l(A) + l(B)$$

(the "property of additivity").

These conditions lead to a unique solution of the problem.

It follows from *(2)* and *(3)* that the qth part of the standard should have the length $1/q$; if this part is repeated p times the obtained segment should have the length p/q, by *(3)*. Thus, if the segment A is commensurable with the standard of length and the common measure of the segments A and E is contained in them p and q times, respectively, then necessarily

$$l(A) = \frac{p}{q}.$$

It is readily seen that this number is independent of the assumed common measure and that if segments commensurable with the standard of length are associated with rational lengths in accordance with this rule, then (for these segments) the problem of measuring is completely solved.

For the general case we see that if segment A is greater than segment B so that $A = B + C$, where C is also a segment, then in view of *(3)* we should have:

$$l(A) = l(B) + l(C)$$

and since $l(C) > 0$, we have $l(A) > l(B)$. Thus, unequal segments should have unequal lengths, and so the greater segment should have a greater length.

Since any positive rational number p/q is the length of a segment commensurable with the standard of length E, it follows incidentally from the above facts that no segment incommensurable with the standard can have a rational length.

Suppose that Σ is such a segment *incommensurable* with E. An infinite number of segments S and S' can be found, commensurable with E and smaller and greater than Σ, respectively. Setting $l(S) = s, l(S') = s'$ we obtain for the required length $l(\Sigma)$ the inequalities

$$s < l\left(\sum\right) < s'\dagger.$$

If all rational numbers be distributed over two classes $\boldsymbol{S}$ and $\boldsymbol{S'}$, placing in the lower class $\boldsymbol{S}$ the numbers s (and, also, all negative numbers and the zero) and in the upper class $\boldsymbol{S'}$ the numbers s', then we have a cut in the set of rational numbers. Since it is evident that in the lower class there is no greatest number and in the upper no smallest number, this cut defines an irrational number σ which is precisely the unique real number which satisfies the inequalities $s < \sigma < s'$. This number will be set equal to the length $l(\Sigma)$.

Assume now that all segments both commensurable and incommensurable with E are associated with lengths in accordance with the rules indicated above. Conditions *(1)* and *(2)* are obviously satisfied.

Consider two segments P and Σ with the lengths

$$\varrho = l(P), \quad \sigma = l(\Sigma)$$

† Obviously, for the length of a segment Σ commensurable with E, the inequalities are also satisfied.

and their sum $T = P + \Sigma$ the length of which is denoted by $\tau = l(T)$. Taking arbitrary positive rational numbers r, r', s, s' such that

$$r < \varrho < r', \quad s < \sigma < s',$$

we construct the segments R, R', S, S' for which these numbers are the lengths. The segment $R + S$ (of length $r + s$) is smaller than T and the segment $R' + S'$ (of length $r' + s'$) is greater than T. Hence,

$$r + s < \tau < r' + s'.$$

But, [Sec. 7], the only real number lying between the numbers of the form $r + s$† and $r' + s'$ is the sum $\varrho + \sigma$. Consequently, $\tau = \varrho + \sigma$ which completes the proof.

The extension of the "property of additivity" to the case of an arbitrary finite number of terms is carried out by the method of mathematical induction.

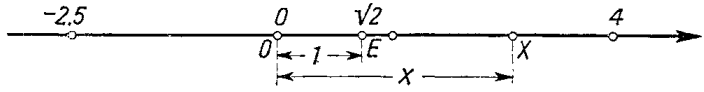

FIG. 1.

If we select on the axis (a directed straight line) (cf. Fig. 1) the original point O and a standard of length E, then there corresponds to every point X of this straight line a real number—with coordinate x; this is equal to the length of the segment OX if X lies in the positive direction from O, and to this length with the minus sign in the other case.

Naturally, the question arises, is the converse also true? Does every real number x correspond in this case to a point of the straight line? This question is answered (in the affirmative) in geometry, with the help of the axiom of continuity of the straight line; this establishes for a straight line, regarded as a set of points, a property analogous to the property of continuity of the set of real numbers [Sec. 5].

Thus, a one-to-one correspondence can be established between all real numbers and points of a directed straight line (an axis). The real numbers can be represented by points on an axis, which hence will be called the number axis. Such a representation will frequently be used below.

† The limitation to positive numbers r and s is, of course, not essential.

CHAPTER 2

FUNCTIONS OF ONE VARIABLE

§ 1. The concept of a function

14. Variable quantity. In investigating natural phenomena and in practical activity man encounters numerous physical quantities; for instance, time, length, volume, velocity, force, mass, etc. Depending on the nature of the problem they can acquire either various values or, alternatively, have only one value. In the first case we call them *variable* quantities and in the second case, *constant* quantities.

If we choose a certain unit of measure (as was done in Sec. 13 for length) any value of a quantity can be expressed by a number. Mathematics is usually not concerned with the physical meaning of the quantities considered, but only with their numerical values. This natural process of abstraction was described by F. Engels† as follows: stating that "the objects of mathematics are the spatial forms and qualitative relations of the real world", he proceeds:

"However, in order to be able to investigate these forms and relations in their natural state, it is necessary to separate them entirely from their contents, leaving the latter aside as irrelevant; thus we obtain points having no dimension, lines devoid of width and thickness, various a and b, x and y, as constant and variable quantities...."

Introducing into mathematics the concept of a variable quantity (this is usually attributed to Descartes) was a step of the greatest importance. Mathematics became capable of not only establishing quantitative relations between constant quantities but also of investigating processes occurring in nature, in which variable quantities could also participate. F. Engels emphasizes this fact in the following words:

"A turning point in mathematics was Descartes' *variable quantity*. Thanks to this, mathematics encompassed motion and dialectic,

† F. Engels. *Dialectic of Nature*, ed. 1952, p. 37 (in Russian).

and owing to this the differential and integral calculus became at once necessary..."†

15. The domain of variation of a variable quantity. In mathematical analysis—providing we do not speak of its applications—by a variable quantity (or briefly a variable) we mean an *abstract* or *numerical* variable. It is denoted by a symbol (a letter, e.g. x) which is endowed with numerical values. The variable x is regarded as prescribed if the set $\mathcal{X} = \{x\}$ of values, which the variable can acquire, is indicated. This set is called the *domain of variation* of the variable x. In general, any numerical set may serve as the domain of variability of a variable.

A constant quantity (briefly a constant) may conveniently be regarded as a particular case of the variable: it corresponds to the assumption that the set $\mathcal{X} = \{x\}$ contains only one element.

We found in Sec. 13 that the numbers have a geometric interpretation as points on an axis. The domain $\mathcal{X}$ of variation of the variable x is represented on this axis as a set of points. Accordingly, usually the numerical values of the variable themselves are called *points.*

Frequently we have to consider a variable n taking all possible positive integral values

$$1, 2, 3, \ldots 100, 101, \ldots;$$

the domain of variation of this variable, i.e. the set $\{n\}$ of positive integers, will always be denoted by $\mathcal{N}$.

However, analysis is usually concerned with variables which vary in a *continuous* manner: they are derived from physical quantities—time, distance covered by a moving point, etc. The domain of variation of such a variable is a numerical *interval.* Most frequently it is a *finite* interval bounded by two real numbers a and b $(a < b)$—its ends may or may not be included in the interval itself. Depending on this we distinguish:

a closed interval $[a, b]$, $a \leqslant x \leqslant b$ (both ends included);

a semi-open interval $\begin{cases} (a, b], \ a < x \leqslant b \\ [a, b), \ a \leqslant x < b \end{cases}$ (only one end included);

an open interval (a, b), $a < x < b$ (neither end included).

† F. Engels. *Dialectic of Nature,* ed. 1952, p. 206 (in Russian).

By the *length* of the interval we always mean the number $b-a$.

The geometric counterpart of the interval is clearly a segment of the numerical axis and, depending on the type of the interval, the end-points may or may not be included in the segment.

Sometimes we have to deal with infinite intervals, one or both ends of which are the "improper" numbers $-\infty, +\infty$. Their notation is analogous to that given above. For instance, $(-\infty, +\infty)$ is the set of all real numbers; $(a, +\infty)$ denotes the set of the numbers x which satisfy the inequality $x>a$; the interval $(-\infty, b]$ is defined by the inequality $x \leqslant b$. Geometrically infinite intervals are represented by the straight line infinite in both directions or by a semi-infinite line.

16. Functional relation between variables. Examples. The object of investigation in mathematical analysis is, however, not the variation of one variable by itself, but the *relation* between two or more variables under their simultaneous variation. Here we shall confine ourselves to the simplest case, that of two variables.

In various domains of science and natural phenomena—in mathematics itself, in physics, engineering—the reader frequently encounters such simultaneously varying quantities. They cannot simultaneously take arbitrary values (from their domains of variation): if one of them (the *independent variable*) is given a definite value, this determines the value of the second (*the dependent variable* or *function*). We give a few examples.

(1). The area Q of the circle is a function of its radius R; its value can be calculated from the value of the radius, by means of the well-known formula

$$Q = \pi R^2.$$

(2). In the case of the free fall of a heavy material point, in the absence of resistance, the time t (in seconds) measured from the beginning of the motion and the distance s (in metres) covered in this time are connected by the relation

$$s = \frac{gt^2}{2},$$

where $g = 9.81$ m/sec^2 is the acceleration due to gravity. Hence the value of s which corresponds to the time t can be determined, i.e. the distance s is a function of the time t.

(3) Consider a mass of a (perfect) gas contained under a piston in a closed cylinder. Assuming that the temperature is constant, the volume V (in litres) and the pressure p (in atmospheres) of this mass of gas obey the Boyle–Mariotte law

$$pV = c = \text{const.}$$

If the volume V be arbitrarily varied, then p as a function of V always takes a uniquely determined value, in accordance with the formula

$$p = \frac{c}{V}.$$

Observe that the very choice of the independent variable from the two under consideration is sometimes arbitrary and made out of simple convenience. However, in most cases the choice is determined by the nature of the investigation. Thus, in the last example we could be interested in the dependence of the volume V on the variable external pressure p on the piston (transferred to the gas); then the formula would naturally be written in the form

$$V = \frac{c}{p},$$

regarding p as the independent variable and V as a function of p.

The functional relation in other cases is characterized by a process which occurs during the passage of time, especially if, as in Example *(2)*, the time itself is the independent variable. However, it would be erroneous to think that the variation of the variables is always connected with the passage of time. In Example *(1)*, when examining the dependence of the area of the circle on the radius we were not dealing with a time process.

17. Definition of the concept of function. We now disregard the physical meaning of the considered quantities, and we present a precise definition of the concept of a *function*—one of the fundamental concepts of mathematical analysis.

Consider two variables x and y with the domains of variation $\mathcal{X}$ and $\mathcal{Y}$. Assume that the conditions of the problem state that the

variable x may take an arbitrary value from the domain $\mathcal{X}$, with no limitations at all. Then the variable y is said to be a function of the variable x in its domain of variation $\mathcal{X}$ if, in accordance with a rule or a law, to every value x from $\mathcal{X}$ there corresponds one definite value of y (from $\mathcal{Y}$).

The independent variable x is also called the *argument* of the function.

In the above definition two things are important: first, the definition of the domain $\mathcal{X}$ of variation of the argument x (it is also called the domain of definition of the function) and, secondly, the establishing of a rule or a law of correspondence between the values of x and y (the domain $\mathcal{Y}$ of the variation of the function y is not usually indicated, since the very law of correspondence determines the set of values taken by the function).

In the definition of the function we may take a more general viewpoint, assuming that to every value of x from $\mathcal{X}$ there corresponds not one but several values of y (and even an infinite set of them). In such cases the function is called *multi-valued*, in contrast to the *single-valued* function defined above. Usually, in courses of analysis based on the study of a real variable, multi-valued functions are avoided and henceforth, when speaking of a function, we shall mean a *single-valued function* unless the contrary is stated.

To indicate the fact that y is a function of x we write

$$y = f(x), \quad y = \varphi(x), \quad y = F(x), \quad \text{etc.}^{\dagger}$$

The letters f, φ, F describe the rule according to which we obtain the value of y corresponding to a prescribed x. Consequently, if at the same time various functions of one argument x are considered, connected with various laws of correspondence, they should not be denoted by the same letter.

Although the letter f (written in various forms) is connected with the word "function", obviously any other letter can be used to denote the functional relation; sometimes even the same letter, y, is repeated: $y = y(x)$. In some cases the argument is written as an index of the function, i.e. y_x.

† This notation is pronounced as follows: "y is equal to f of x", "y is equal to φ of x", etc.

If considering a function, say $y = f(x)$, we may want to indicate its particular value, equal to x_0; to denote this we use the symbol $f(x_0)$. For instance, if

$$f(x) = \frac{1}{1+x^2}, \quad g(t) = \frac{10}{t}, \quad h(u) = \sqrt{(1-u^2)},$$

then $f(1)$ denotes the numerical value of the function $f(x)$ when $x = 1$, i.e. simply the number $1/2$; similarly, $g(5)$ denotes the number 2, $h(3/5)$ the number $4/5$, etc.

Let us now turn to the rule or the law of correspondence between the values of the variables; it is this which constitutes the essence of the concept of a functional relation.

The most simple and natural way is to realize this rule by means of a formula which represents the function in the form of an analytic expression indicating the analytic operations over the real numbers and the values of x which are to be carried out in order to obtain the corresponding value of y. This *analytic method* of prescribing a function is most important for mathematical analysis (we shall return to it later). The reader is already familiar with this concept from school courses of mathematics; it was the analytic method that was employed in the examples given in Sec. 16.

Nevertheless it would be erroneous to think that this is the only method of prescribing a function. In mathematics itself we quite frequently have cases when the function is given without a formula: for instance, the function $E(x)$—"the integral part of the number x"†. It is readily observed that

$$E(1) = 1, \quad E(2.5) = 2, \quad E(\sqrt{13}) = 3, \quad E(-\pi) = -4, \quad \text{etc.},$$

although there is no formula representing $E(x)$.

In science and in engineering the relation between quantities is frequently established by means of an experiment or by observations. For instance, if water be subjected to an arbitrary pressure p (in atmospheres) then, experimentally, we can determine the corresponding temperature θ°C at which boiling occurs: θ is a function of p. However, this functional relation is not given by any formula but simply

† More precisely, the greatest integer not exceeding x. (E is the first letter of the French word *entier* meaning "entire" or "integral".)

by a table containing the data obtained from experiments. Examples of the tabular method of prescribing a function can easily be found in any engineering handbook. The inconvenience consists in the fact that the method gives values of the function only for certain values of the argument.

Let us finally mention that in some cases, by means of self-recording instruments, the functional relation between physical quantities is given directly by a graph. For instance, the "indicator diagram" taken by the indicator gives the relation between the volume V and the pressure p of steam in a cylinder of a working steam engine; the "barogram" supplied by the barograph represents the daily variation of atmospheric pressure, etc. Of course, this way of prescribing a function determines its values only approximately.

We do not consider the details of the tabular and graphical methods of prescribing a functional relation, since they are not used in mathematical analysis.

18. Analytic method of prescribing a function. We now make a few explanatory remarks on the method of prescribing a function by an *analytic expression* or a *formula*; this plays an exceptional part in mathematical analysis.

(1) First of all we consider what *analytic operations* may enter the formulae. We, of course, mean here the operations investigated in elementary algebra and trigonometry: viz., the arithmetical operations, raising to a power (and taking a root), finding logarithms, transition from the angles to the trigonometric quantities and conversely (see below, § 2 of this chapter). However, it is important to note that in the course of the progress of our knowledge of analysis, other operations will be added to the above ones, e.g. the passage to a limit, to which Chapter 3 is devoted.

In this way, the complete meaning of the phrase "analytic expression" or "formula" will gradually be disclosed.

(2) The second remark concerns the *domain of definition* of a function by an analytic expression or a formula.

Any analytic expression containing the argument x has in a way a natural domain of application; this is the set of all values of x for which it has a meaning, i.e. a fully determined, finite, real value. Let us elucidate this statement by simple examples.

Thus, for the expression $1/(1+x^2)$ this domain is the whole set of real numbers. For the expression $\sqrt{(1-x^2)}$ this domain reduces to the closed interval $[-1, 1]$ outside the limits of which the value of $\sqrt{(1-x^2)}$ is no longer real. However, for the expression $1/\sqrt{(1-x^2)}$ we have to take as the natural domain the open interval $(-1, 1)$, since at its end-points the dominator vanishes. Sometimes, the domain of values for which the expression has a meaning consists of separate intervals: for $\sqrt{(x^2-1)}$ these are the intervals $(-\infty, -1]$ and $[1, +\infty)$, for $1/(x^2-1)$ the intervals $(-\infty, -1)$, $(-1, 1)$ and $(1, +\infty)$, etc.†.

In the remainder of this book we shall have to consider more complicated and more general analytic expressions and frequently we shall investigate properties of functions prescribed by such an expression in the whole domain where it has meaning, i.e. we shall consider the *mathematical apparatus* itself.

However, we draw the reader's attention to a different situation which can arise. Imagine that some definite problem in which the variable x is naturally confined to the domain of variation $\mathfrak{X}$ has led us to consider a function $f(x)$ having an analytic expression. Although it may happen that this expression also has meaning outside the domain $\mathfrak{X}$, in this problem x cannot take values outside $\mathfrak{X}$; here the analytic expression has an auxiliary value.

For instance, if examining the free fall of a heavy point from a height h above the surface of the earth we use the formula

$$s = \frac{gt^2}{2}$$

[Sec. 16, *(2)*], it would be meaningless to consider negative values of t or values of t greater than $T = \sqrt{(2h/g)}$, since, as easily observed, for $t = T$ the point falls on the ground. This is true although the expression $gt^2/2$ itself is valid for all real t.

(3) It may happen that the function is prescribed by more than one formula for different values of the argument, one formula being used for some of its values and a different one for other values.

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† Evidently, we are not interested in expressions which have no meaning for any value of x.

An example of such a function in the interval $(-\infty, +\infty)$ is provided by the function given by the following three formulae:

$$f(x) = \begin{cases} 1 & \text{if} \quad |x| > 1 \quad \text{(i.e. if } x > 1 \text{ or } x < -1\text{)}, \\ -1 & \text{if} \quad |x| < 1 \quad \text{(i.e. if } -1 < x < 1\text{)}, \end{cases}$$

and finally,

$$f(x) = 0 \quad \text{if} \quad x = \pm 1.$$

Observe that there is no essential difference between a function prescribed by one formula for all values of x and a function the definition of which requires several formulae. Usually, a function prescribed by several formulae (at the expense of complicating the expression) may also be given by one formula. In particular, this is true for the above function (see Sec. 43, *(5)*). In what follows we shall frequently encounter such examples.

19. Graph of a function. Although in mathematical analysis functions are not given graphically, graphical illustration is often used. Clearness and visual demonstration of a graph render it an indispensable auxiliary device for investigating properties of functions.

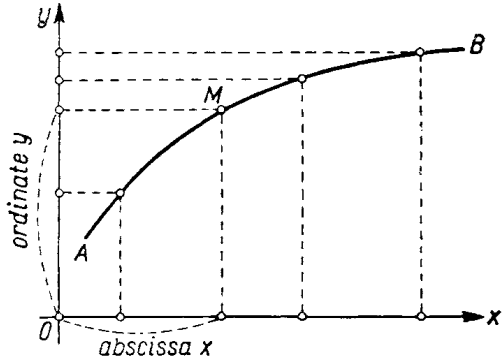

FIG. 2.

Let a function $y = f(x)$ be given in an interval $\mathfrak{X}$. Construct in a given plane two perpendicular coordinate axes, the x-axis and the y-axis. Consider the pair of values x and y, x being taken from the interval $\mathfrak{X}$ and $y = f(x)$; the image of this pair on the plane is the point $M(x, y)$ with the abscissa x and the ordinate y. The set of such points obtained in varying x inside its interval constitutes the graph of the function, i.e. the geometric image of the function.

Usually the graph constitutes a curve similar to AB in Fig. 2. The equation $y = f(x)$ itself is then called the *equation of the curve AB.*

For instance, Figs. 3 and 4 represent the graphs of the functions

$$y = \pm\sqrt{(1-x^2)} \quad (|x| \leqslant 1) \qquad \text{and} \qquad y = \pm\sqrt{(x^2-1)} \quad (|x| \geqslant 1);$$

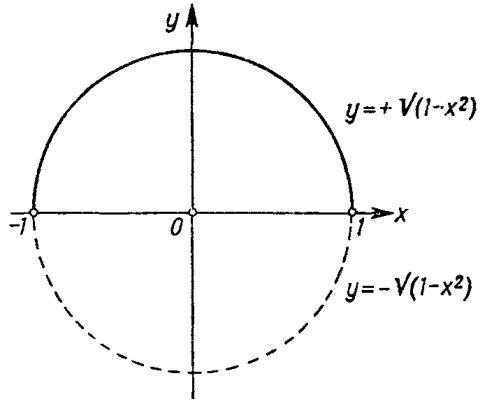

FIG. 3.

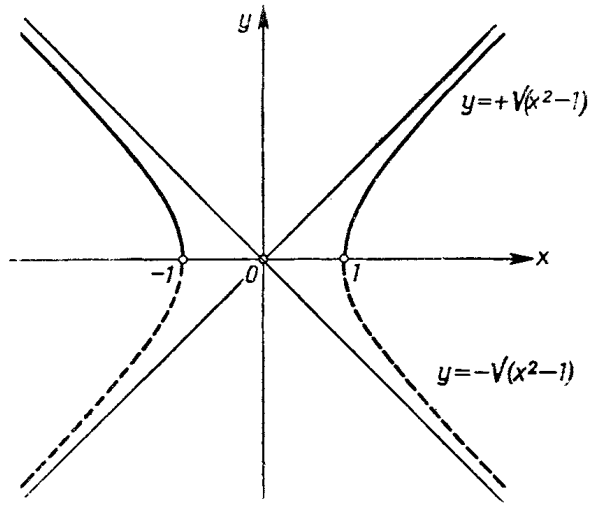

FIG. 4.

the reader recognizes the circle and rectangular hyperbola. Numerous examples of graphical representation may be found in the following subsections.

Graphs are usually constructed by means of points. We take in the interval $\mathfrak{X}$ a number of values of x close together and we calculate by means of the formula the corresponding values of $y = f(x)$:

$$\frac{x = x_1 | x_2 | \ldots | x_n}{y = y_1 | y_2 | \ldots | y_n},$$

then we indicate on the graph page the points

$$(x_1, y_1),\ (x_2, y_2),\ \ldots,\ (x_n, y_n).$$

Through these points we draw a curve which gives (of course, approximately) the required graph. The smoother the curve and the closer the points are taken, the more exactly the drawn curve represents the graph.

It should be observed that although the geometric image of the function can always be constructed, this image will not always be a curve in the ordinary sense of the word.

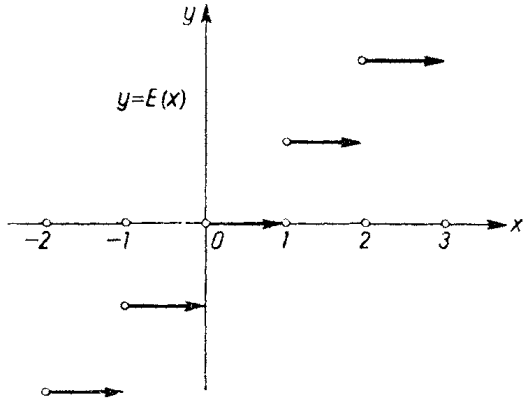

FIG. 5.

Let us for instance construct the graph of the function $y = E(x)$. Since in the intervals $\ldots, [-2, -1), [-1, 0), [0, 1), [1, 2), [2, 3), \ldots$ the function has constant values $\ldots, -2, -1, 0, 1, 2, \ldots$, the graph consists of a number of separate horizontal segments without their right-hand ends (Fig. 5)†.

† This fact is indicated by arrows which point towards the points not belonging to the graph.

20. Functions of positive integral argument. So far we have considered only examples of functions of a continuously varying argument the values of which filled a continuous interval. Let us now consider a basically simpler (but not less important) case of a function $f(n)$ of the argument n taking on only the values of the set of positive integers $\mathcal{N}$. *The functions of positive integral argument will play a special role in what follows.*

In denoting such a function we frequently abandon the ordinary notation and instead of $f(n)$ we write any letter with the index n below, for instance x_n. If this index be replaced by a definite positive integer (remembering that it is an independent variable), say 1, 23, 518, ..., then $x_1, x_{23}, x_{518}, \ldots$ are the corresponding numerical values of the function x_n, just as $f(1), f(23), f(518), \ldots$ denote the numerical values of the function $f(n)$.

In accordance with the general definition the function x_n is regarded as known if we know a rule according to which any of its values can be determined for arbitrary n.

The ordinary case occurs when the function x_n is given by a formula establishing what analytic operations have to be performed on the positive integer n (and on the constants) in order to obtain the corresponding value of the function. For example,

$$x_n = \frac{n^2 - n + 2}{3n^2 + 2n - 4}, \quad a_n = q^n, \quad y_n = \log n, \quad \text{etc.}$$

However, it is evident that the function can be prescribed by any other rule. As an example consider the "factorial of the number n"

$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n,$$

and the function $\tau(n)$ representing the number of divisors of the number n, or the function $\varphi(n)$ indicating the number of relatively prime numbers in the sequence $1, 2, 3, \ldots, n$. In spite of the peculiar nature of the rules by which these functions are prescribed, they make it possible to calculate values of functions with the same definiteness as if explicit formulae were known:

$$\tau(10) = 4, \quad \tau(12) = 6, \quad \tau(16) = 5, \quad \ldots$$

$$\varphi(10) = 4, \quad \varphi(12) = 4, \quad \varphi(16) = 8, \quad \ldots.$$

Another example is as follows: let us represent the decimal approximations for $\sqrt{2}$, with increasing accuracy

$$1.4;\ 1.41;\ 1.414;\ 1.4142;\ \ldots.$$

Knowing the rule for an approximate calculation of the roots we can regard as fully determined the function defined as the approximate value of the above root with accuracy $1/10^n$, although we have no general expression for this approximation.

In school courses of mathematics the reader frequently encountered functions of a positive integral index. If we are given the infinite geometric progression

$$\therefore a,\ aq,\ aq^2,\ \ldots,$$

the function of the index n is also the general term of this progression

$$a_n = aq^{n-1}$$

and the sum of n terms of the progression is

$$s_n = \frac{a - aq^n}{1 - q} \qquad (q \neq 1).$$

In defining the circumference and the area of a circle, one usually considers regular polygons inscribed in the circle; these are obtained from an inscribed hexagon by consecutive doubling of number of sides.

A side of such a polygon, its perimeter and area are all functions of the positive integral index n, if for n we simply take the number of times we have doubled the number of the sides.

21. Historical remarks. The very term "function" appeared in one of Leibniz's papers† in 1692 and later was applied by the brothers Jacob and Johann Bernoulli‡ to describe various segments in some way connected with points of a curve. In 1718 John Bernoulli first announced a definition (of a

† Gottfried Wilhelm Leibniz (1646–1716)—an outstanding German philosopher and mathematician. He shares with Newton the credit for the creation of the differential and integral calculus (see historical review in Chapter 14).

‡ Jacob Bernoulli (1654–1705) and John Bernoulli (1667–1748) belonged to a family of Dutch origin which was outstanding in the history of mathematics; they both were associates of Leibniz and contributed greatly (particularly the younger one) to the dissemination of the new calculus.

function which was free of geometric representations. His pupil Euler† in his manual *Introduction to the Analysis of Infinitesimals* (1748), which was a textbook for many generations of mathematicians, reproduces Bernoulli's definition in a somewhat more precise way:

"A function of a variable quantity is an analytic expression constructed in some manner from this variable quantity and from numbers or constant quantities."‡

We observe that in this definition the function is identical with the analytic expression by which it is prescribed.

Besides "explicit" functions Euler considered also "implicit" functions defined by insoluble equations. At the same time in connection with the celebrated problem of the vibration of a string (we shall consider it in detail in the second volume) Euler thought it possible to introduce into analysis not only "mixed" functions which, in various parts of the interval, are given by various analytic expressions (cf. Sec. 18, (3)), but even functions defined by graphs drawn in an arbitrary manner. In the foreword to his *Differential calculus* (1755) we encounter the even more general, although less definite, formulation:

"When certain quantities depend on others in such a way that in varying the latter they are also subject to a variation, then the former are said to be functions of the latter ones."§

For many decades there was no essential progress in the definition of the concept of function. Usually Dirichlet is said to have the credit†† of emphasizing the notion of correspondence which is the only basis of this concept.

In 1837 he announced the following definition of the function y of the variable x (under the assumption that the latter takes all values in a certain interval):

"If to every x there corresponds a unique finite y, then y is said to be a function of x for this interval. Then it is entirely unnecessary that y depends on x according to one law in the whole interval, and moreover it is even unnecessary to imagine a relation expressed by means of mathematical operations."

This definition played an important role in the history of mathematical analysis.

For a long time it went unnoticed that Lobatchevsky‡‡ announced this idea not only earlier but in an irreproachable manner. Agreeing first with Euler's

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† Leonhard Euler (1707–1783)—an outstanding mathematician; he was of Swiss origin, spent the greater part of his working life in Russia and was a member of the St. Petersburg Academy of Sciences.

‡ There is a Russian translation of vol. 1 (originally written in Latin), 1936; p. 30.

§ See the Russian translation of *Differential Calculus*, 1949, p. 38.

†† Peter Gustave Lejeune Dirichlet (1805–1859)—an outstanding German mathematician.

‡‡ Nikolai Ivanovitch Lobatchevsky (1793–1856)—a great Russian mathematician, famous for creating non-Euclidean geometry.

standpoint Lobatchevsky gradually abandons it and in his paper "On the Vanishing of Trigonometric Lines" (1834) he states:

"A general definition requires that we call the function of x, the number which is given for every x and which gradually varies with x. The value of a function can be given either by an analytic expression or by a condition which supplies us with a method of examination of all numbers and choosing one of them; or, finally, the relation can exist and remain unknown."†

Let us finally observe that the customary notation of function, $f(x)$, is due to Euler.

§ 2. Important classes of functions

22. Elementary functions. Let us enumerate some classes of functions which are called elementary.

(1) Integral and fractional rational functions. A function represented by the polynomial in x

$$y = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$$

($a_0, a_1, a_2, \ldots$ are constants) is called an *integral rational function.*

The ratio of two such polynomials

$$y = \frac{a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n}{b_0x^m + b_1x^{m-1} + \ldots + b_{m-1}x + b_m}$$

is called a *fractional rational function.* It is defined for all values of x except those for which the denominator vanishes.

For instance, Fig. 6 shows the graphs of the function $y = ax^2$ (parabola) for various values of the coefficient a and Fig. 7 the graphs of the function $y = a/x$ (rectangular hyperbola), again for various values of a.

(2) Power functions. This is the function of the form

$$y = x^\mu,$$

where μ is an arbitrary constant number. For an integral μ we obtain a rational function. For a fractional μ we have a root. For instance, let m be a positive integer and

$$y = x^{\frac{1}{m}} = \sqrt[m]{x}.$$

This function is defined for all values of x if m is odd and only for non-negative values of x if m is even (in this case we mean the

† N. I. Lobatchevsky, *Complete Works*, vol. V (1951), p. 43 (in Russian).

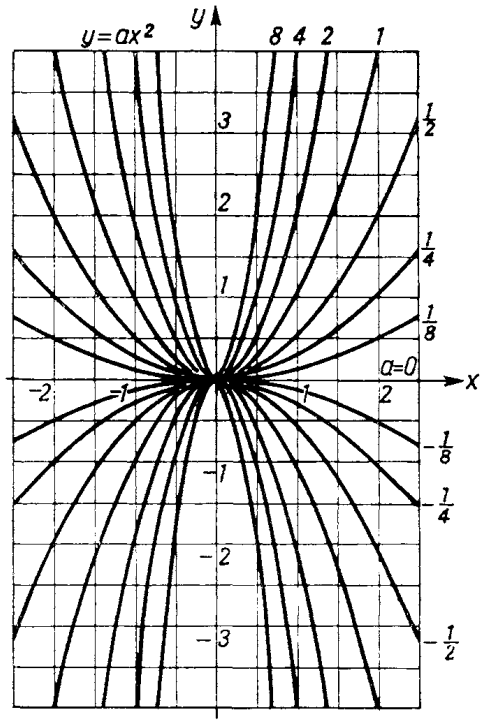
y
y=ax²
8 4 2 1
3
2
1
1/2
1/4
1/8
a=0
-2
-1
1
2
x
-1/8
-1/4
-1
-2
-3
-1/2

FIG. 6.

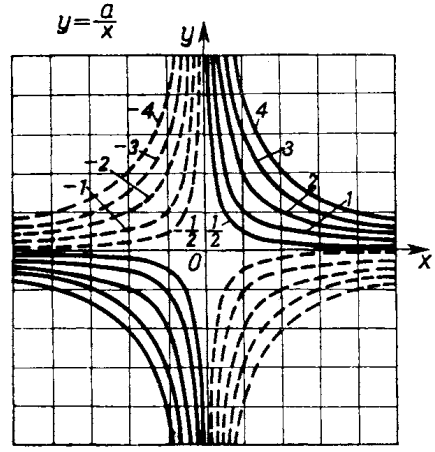
y=a/x
y
-4
-3
-2
-1
4
3
2
1
1/2
1/2
0
x

FIG. 7.

arithmetical value of the root). Finally, if μ is an irrational number we assume that $x > 0$ ($x = 0$ is allowed only for $\mu > 0$).

Figures 8 and 9 show the graphs of the power function for various values of μ.

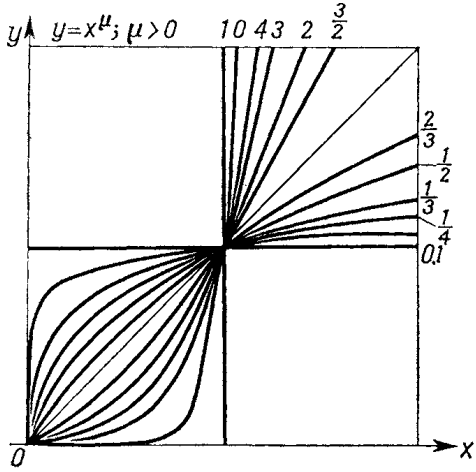

FIG. 8.

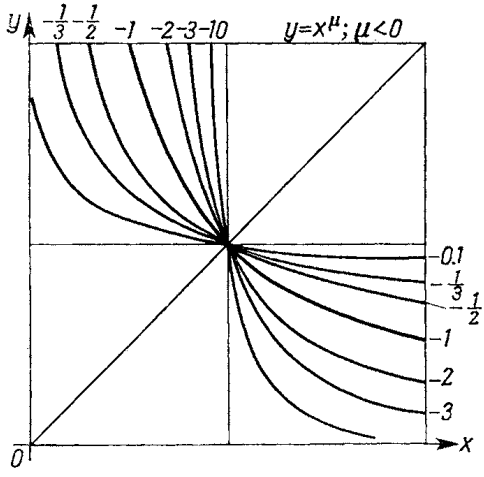

FIG. 9.

(3) Exponential functions. That is, functions of the form

$$y = a^x,$$

where a is a positive number different from unity; x takes any real value.

The graphs of the exponential function for various values of a are given in Fig. 10.

(4) Logarithmic functions. That is, functions of the form

$$y = \log_a x^{\dagger},$$

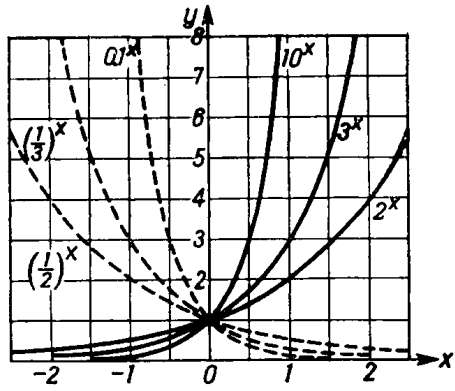

FIG. 10.

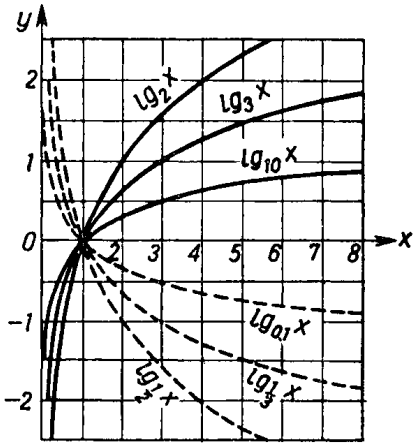

FIG. 11.

where a as before is a positive number (different from unity); x takes only positive values.

In Fig. 11, graphs of this function are given for various values of a.

† In the translation we have used log x to denote $\log_e x$. We shall not drop the suffix 10 of $\log_{10} x$.

(5) Trigonometric functions.

$$y = \sin x, \quad y = \cos x, \quad y = \tan x, \quad y = \cot x,$$
$$y = \sec x, \quad y = \operatorname{cosec} x.$$

It is important always to remember that the arguments of trigonometric functions, if they are regarded as measures of angles, always represent these angles in radians (unless the contrary is stated). For $\tan x$ and $\sec x$, values of the form $(2k+1)\pi/2$ are excluded, while for $\cot x$ and $\operatorname{cosec} x$ the values of the form $k\pi$ (k is an integer) are excluded.

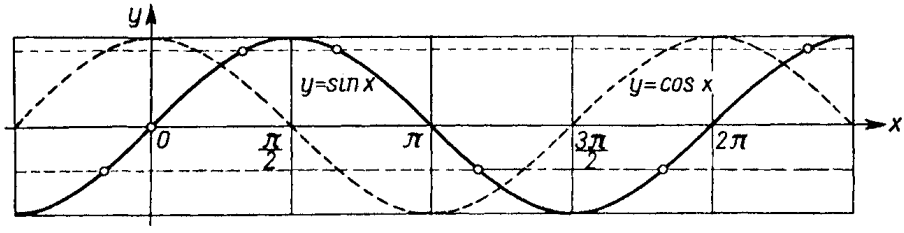

FIG. 12.

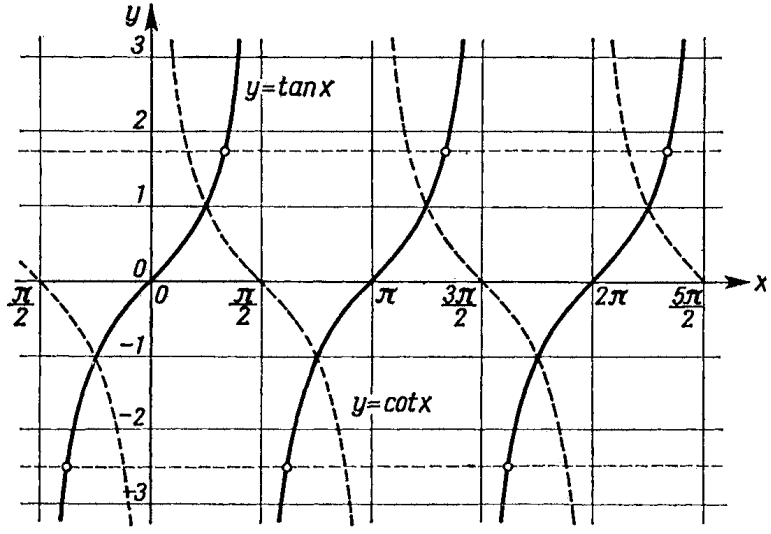

FIG. 13.

The graphs of the functions $y = \sin x (\cos x)$ and $y = \tan x (\cot x)$ are given in Figs. 12 and 13. The graph of the sine is usually called the *sinusoid.*

23. The concept of the inverse function. Before proceeding to inverse trigonometric functions let us make a remark about inverse functions in general.

Assume that the function $y = f(x)$ is given in a domain $\mathcal{X}$ and let $\mathcal{Y}$ be the set of all values which this function takes when x ranges in the domain $\mathcal{X}$. In our case both $\mathcal{X}$ and $\mathcal{Y}$ represent intervals.

Select any value $y = y_0$ from the domain $\mathcal{Y}$; then in the domain $\mathcal{X}$ a value $x = x_0$ can always be found for which our function takes the value y_0, i.e.

$$f(x_0) = y_0;$$

there can be a number of such values of x_0. Thus, every value y from $\mathcal{Y}$ is associated with one or more values of x; this defines in the domain $\mathcal{Y}$ a *single-valued* or *multi-valued* function $x = g(y)$ which is called the inverse of the function $y = f(x)$.

Let us consider some examples.

1. Let $y = a^x (a > 1)$ where x varies over the interval $\mathcal{X} = (-\infty, +\infty)$. The values of y fill the interval $\mathcal{Y} = (0, +\infty)$ and to every y from this interval there corresponds, as we know, [Sec. 12], one definite value $x = \log_a y$ in $\mathcal{X}$. In this case the inverse function is *single-valued.*

2. On the contrary, for the function $y = x^2$, if x varies over the interval $\mathcal{X} = (-\infty, +\infty)$, the inverse function is *two-valued*; to every value y from the interval $\mathcal{Y} = [0, +\infty)$ there correspond two values of $x = \pm\sqrt{y}$ from $\mathcal{X}$. Instead of this two-valued function one usually considers separately two single-valued functions $x = +\sqrt{y}$ and $x = -\sqrt{y}$ ("branches" of the two-valued function). They can also be regarded separately as inverse to the function $y = x^2$, assuming only that the domain of variation of x is bounded by the intervals $[0, +\infty)$ and $(-\infty, 0]$, respectively.

Observe that the graph of the function $y = f(x)$ clearly indicates whether the inverse function $x = g(y)$ is single-valued or not. The first case occurs if any straight line parallel to the x-axis cuts the graph only at one point. On the contrary, if some of these straight lines cut the graph at several points, the inverse function is multi-valued. In this case, in accordance with the graph, it is easy to split up the interval of variation of x into parts so that to every part

there corresponds only one "branch" of the function. For instance, from a first glance at the parabola in Fig. 14, which represents the graph of the function $y = x^2$, it is clear that the inverse function is two-valued and to obtain single-valued "branches" it is sufficient to consider separately the right-hand and the left-hand sides of the parabola, i.e. the positive and negative values of x†.

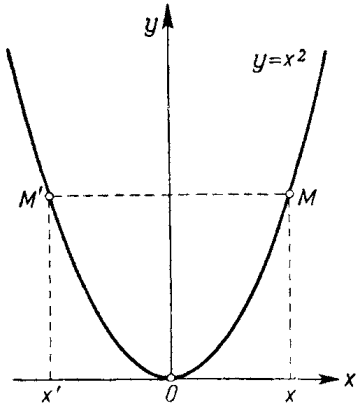

FIG. 14.

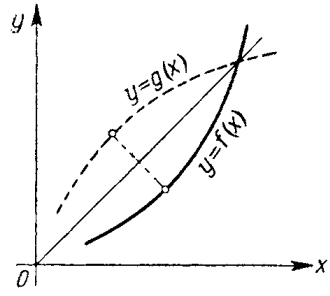

FIG. 15.

If the function $x = g(y)$ is the inverse of the function $y = f(x)$, then it is evident that the graphs of the two functions coincide. However, we can also denote the argument of the inverse function by the letter x, i.e. instead of the function $x = g(y)$ we can write $y = g(x)$. Then we only have to call the horizontal axis the y-axis and the vertical axis the x-axis, the graph remaining unaltered. If we wish the (new) x-axis to be horizontal and the (new) y-axis to be vertical, these axes should be exchanged, thus altering the graph. To do this we simply turn the xOy plane through 180° about the bisector of the first quadrant (Fig. 15).

Thus, finally, the graph $y = g(x)$ is obtained as the mirror image of the graph $y = f(x)$ with respect to this bisector. For instance, it is clear from Figs. 10 and 11 that they can be obtained directly one from the other. Analogously, on the basis of the above reasoning

† Below, [Sec. 71], we shall return to the problem of the existence and single-valuedness of an inverse function.

it is easy to explain the symmetry (with respect to the bisector) of each of Figs. 8 and 9.

24. Inverse trigonometric functions. In addition to the classes of elementary functions mentioned in Sec. 22 we now consider:

(6) Inverse trigonometric functions.

$$y = \operatorname{arc\,sin} x, \quad y = \operatorname{arc\,cos} x, \quad y = \operatorname{arc\,tan} x,$$
$$y = \operatorname{arc\,cot} x, \quad (y = \operatorname{arc\,sec} x, \quad y = \operatorname{arc\,cosec} x).$$

We examine the first. The function $y = \sin x$ is defined in the interval $\mathcal{X} = (-\infty, +\infty)$ and its values fill continuously the interval $\mathcal{Y} = [-1, 1]$. A line parallel to the x-axis cuts the sinusoid, i.e. the graph of the function $y = \sin x$ (Fig. 12), at an infinite set of points; in other words, to every value of y from the interval $[-1, 1]$ there corresponds an infinite set of values of x. Therefore, the inverse function which we denote by

$$x = \operatorname{Arc\,sin} y^{\dagger}$$

is infinitely valued.

Usually only one "branch" of this function is considered—that which corresponds to x varying between $-\pi/2$ and $+\pi/2$. To every y from $[-1, 1]$ in these bounds there corresponds one value of x; it is denoted by

$$x = \operatorname{arc\,sin} y$$

and is called the *principal value* of the inverse sine.

Turning now the sinusoid about the bisector of the first quadrant (Fig. 16) we obtain the graph of a multi-valued function $y = \operatorname{Arc\,sin} x$; the graph of its *principal branch* $y = \operatorname{arc\,sin} x$ is drawn in bold line, and it is *single-valued* for x in the interval $[-1, 1]$ and therefore it satisfies the inequality

$$-\frac{\pi}{2} \leqslant \operatorname{arc\,sin} x \leqslant \frac{\pi}{2},$$

distinguishing it from the other branches.

---

† We have already indicated [Sec. 22, *(5)*] that the argument x of the trigonometric function expresses the angle in *radians*; obviously, here also, if we consider the values of the inverse trigonometric functions as measures of angles they are given in *radians*.

Recalling from elementary trigonometry the expressions for an angle in terms of its given sine by one of the values of the latter, it is easy to write down formulae yielding all values of the inverse sine,

$$\operatorname{Arc}\sin x = \arcsin x + 2k\pi \quad \text{or} = (2k+1)\pi - \arcsin x$$

$$(k = 0, \pm 1, \pm 2, \ldots).$$

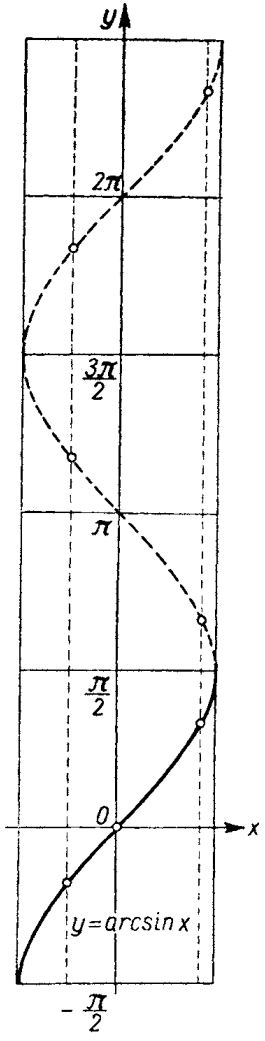

FIG. 16.

Similar reasoning can be applied to the function $y = \cos x$ $(-\infty < x < +\infty)$. Here again the inverse function

$$y = \operatorname{Arc}\cos x \qquad (-1 \leqslant x \leqslant 1)$$

turns out to be infinitely valued (see Fig. 12). To separate the single-valued branch we subject the function to the condition

$$0 \leqslant \operatorname{arc}\cos x \leqslant \pi;$$

this is the *principal branch* of the inverse cosine.

The function $\operatorname{arc}\cos x$ is connected with $\operatorname{arc}\sin x$ by the obvious relation

$$\operatorname{arc}\cos x = \frac{\pi}{2} - \operatorname{arc}\sin x;$$

in fact, not only the cosine of the angle $(\pi/2) - \operatorname{arc}\sin x$ is equal to $\sin(\operatorname{arc}\sin x) = x$ but also the angle itself varies between 0 and π. The remaining values of $\operatorname{Arc}\cos x$ are expressed by the principal value in accordance with the formula

$$\operatorname{Arc}\cos x = 2k\pi \pm \operatorname{arc}\cos x \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

The function $y = \tan x$ is defined for all values of x except for the values

$$x = (2k+1)\frac{\pi}{2} \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

The values of y fill the interval $(-\infty, +\infty)$ and to every value of y there again corresponds an infinite set of values of x (see Fig. 13). Consequently, the inverse function $x = \operatorname{Arctan} y$ given in the interval $(-\infty, +\infty)$ is infinitely valued. In Fig. 17 the graph of the function $y = \operatorname{Arctan} x$ is obtained by turning function $y = \tan x$ through 180° about the bisector of the first quadrant. As the principal value of the inverse tangent $\operatorname{arctan} x$ we take the value of this multi-valued function such that

$$-\frac{\pi}{2} < \operatorname{arc}\tan x < \frac{\pi}{2}.$$

Thus we define the single-valued function—the *principal branch* of the inverse tangent, defined for all values of x. The remaining values of the inverse tangent can easily be shown to be the following:

$$\operatorname{Arc}\tan x = \operatorname{arc}\tan x + k\pi \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

It is easy to establish a direct relation between the functions $\arctan x$ and $\arcsin x$:

$$\underset{(-\infty<x<+\infty)}{\arctan x} = \arcsin \frac{x}{\sqrt{(1+x^2)}} \quad \text{or} \quad \underset{(-1<x<+1)}{\arcsin x} = \arctan \frac{x}{\sqrt{(1-x^2)}}.$$

For instance, if we set $\alpha = \arctan x$ so that $\tan\alpha = x$, then $\sin\alpha = \tan\alpha/\sqrt{(1+\tan^2\alpha)} = x/\sqrt{(1+x^2)}$, the root being taken with the plus sign, since $-\pi/2 < \alpha < \pi/2$; this implies that $\alpha = \arcsin x/\sqrt{(1+x^2)}$.

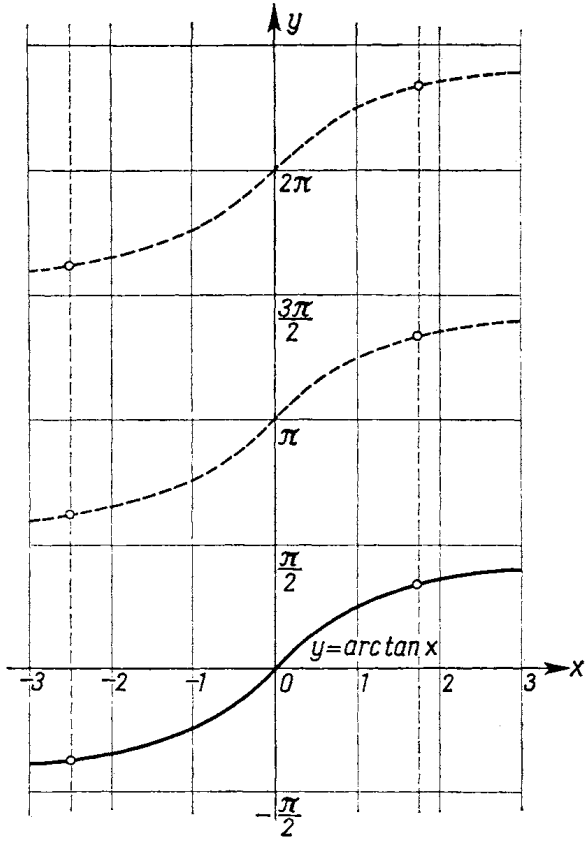

FIG. 17.

Let us also mention the function $\operatorname{Arccot} x$ $(-\infty < x < +\infty)$; its *principal value* is defined by the inequalities

$$0 < \operatorname{arc\,cot} x < \pi$$

and it is connected with $\arctan x$ by the relation

$$\operatorname{arc\,cot} x = \frac{\pi}{2} - \arctan x.$$

The remaining values of the inverse cotangent have the form

$$\operatorname{Arc\,cot} x = \operatorname{arc\,cot} x + k\pi \quad (k = 0, \pm 1, \pm 2, \ldots).$$

We shall not consider the functions $\operatorname{arcsec} x$ ($-\infty < x \leqslant -1$ and $1 \leqslant x < +\infty$) and $\operatorname{arccosec} x$ (with the same intervals of variation), leaving this to the reader.

25. Superposition of functions. Concluding remarks. We shall now introduce the concept of superposition of functions which consists in replacing the argument of a given function by another function (of another argument). For instance, the superposition of the functions $y = \sin x$ and $z = \log y$ yields the function $z = \log \sin x$; similarly we obtain the functions

$$\sqrt{(1 - x^2)}, \quad \arctan \frac{1}{x}, \quad \text{etc.}$$

In general assume that the function $z = \varphi(y)$ is defined in a domain $\mathcal{Y} = \{y\}$ and the function $y = f(x)$ is defined for x in the domain $\mathcal{X} = \{x\}$, *its values lying in the domain* $\mathcal{Y}$. Then it is said that the variable z via y is itself a function of x:

$$z = \varphi(f(x)).$$

Taking x from $\mathcal{X}$ we first find the corresponding value of y from $\mathcal{Y}$ (in accordance with the rule prescribed by the sign f) and next we find the value of z corresponding to this value of y (in accordance with the rule prescribed by the sign φ); the latter is regarded as corresponding to the selected value of x. The obtained function of a function or implicit function is the result of superposition of the functions $f(x)$ and $\varphi(y)$.

The assumption that the values of the function $f(x)$ do not leave the domain $\mathcal{Y}$ in which the function $\varphi(y)$ is defined is *essential.* For instance, setting $z = \log y$ and $y = \sin x$ we may consider only such values of x for which $\sin x > 0$, for otherwise the expression $\log \sin x$ would be meaningless.

It seems to us advantageous to emphasize here that the description of a function as implicit is connected not with the nature of the functional dependence of z on x but only with the manner of prescribing this dependence. For instance, let $z = \sqrt{(1-y^2)}$ for y in $[-1, 1]$ and $y = \sin x$ for x in $[-\pi/2, \pi/2]$. Then,

$$z = \sqrt{(1-\sin^2 x)} = \cos x.$$

Here the function $\cos x$ turns to be prescribed as an implicit function.

Now, having completely explained the concept of superposition we can describe the simplest class of functions encountered in analysis: this contains first of all the above considered elementary functions *(1)-(6)* and then all these functions which may be derived from them by the four arithmetical operations and the superposition, applied consecutively a finite number of times. It is said that they *are expressible by elementary functions in a finite form*; sometimes they are also called *elementary.*

Later on, having mastered more complicated mathematical apparatuses (infinite series, integrals), we shall become acquainted with other functions which also play a great role in analysis but which are at present outside the class of elementary functions.

CHAPTER 3

THEORY OF LIMITS

§ 1. The limit of a function

26. Historical remarks. The concept of limit now enters into the whole of mathematical analysis and also plays an important part in other branches of mathematics. However (as the reader will see in Chapter 14), this concept was certainly not the basis of the differential and integral calculus at the time of their creation. The concept of a limit appears for the first time (essentially in the same form as it will be given below in Sec. 28) in the works of Wallis† in his *Arithmetic of Infinite Quantities* (1655). Newton in the celebrated *Mathematical Foundations of Natural Philosophy* (1686–1687) announced his method of the first and last ratios (sums) in which the beginnings of the theory of limits can be seen. However, none of the great mathematicians of the eighteenth century tried to base the new calculus on the concept of limit and by doing so to meet the just criticism to which the calculus was subject‡. In this respect Euler's views are characteristic; in the foreword to his treatise on *Differential Calculus* (1755) he clearly speaks of the limit but nowhere in the book makes use of this concept.

The turning point in this problem is due to the *Algebraic Analysis* (1821) of Cauchy§ and his further publications, in which for the first time the theory of limits was developed; it was used by Cauchy as an effective means to a precise construction of mathematical analysis. Cauchy's standpoint, which destroyed the mystique surrounding the foundations of analysis, was widely recognized.

Strictly speaking, Cauchy's merit is shared also by other scholars—particularly Bolzano; in many cases his papers were prior to those of Cauchy and later mathematicians. They however were not known at the time and were remembered only after many decades.

27. Numerical sequence. The establishing of the basic concept of analysis, the limit, will be begun by considering the simplest example (known already from the school course), namely the limit of the function x_n of a positive integral argument. We shall see that all the more complicated cases reduce, in principle, to this one.

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† John Wallis (1616–1703)—an English mathematician.

‡ For more detail see Chapter 14.

§ Augustin Louis Cauchy (1789–1857)—an outstanding French analyst.

The argument n takes in turn all the values of the *integral sequence*

$$1, 2, 3, \ldots, n, \ldots, n', \ldots, \tag{1}$$

the terms of which we represent as ordered and increasing, so that the greater number n' follows the smaller number n, while the smaller number n precedes the greater number n'.

If a function x_n is given, its argument or index n can be regarded as the number of the corresponding value of the variable. Thus, x_1 is its first value, x_2 the second, x_3 the third and so on. We shall always represent this set of values $\{x_n\}$ as *ordered, corresponding to the integral sequence* (1), i.e. as the numerical sequence

$$x_1, x_2, x_3, \ldots, x_n, \ldots, x_{n'}, \ldots^{\dagger}. \tag{2}$$

As $n' > n$ the value $x_{n'}$ follows x_n (x_n precedes $x_{n'}$) no matter whether $x_{n'}$ itself is greater or smaller or even equal to x_n.

For instance, if the function x_n be given by one of the formulae

$$x_n = 1, \quad x_n = (-1)^{n+1}, \quad x_n = \frac{1+(-1)^n}{n},$$

the corresponding sequences are the following:

$$\begin{array}{cccccccc} 1, & 1, & 1, & 1, & 1, & 1, \ldots, \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

$$\begin{array}{cccccccc} 1, & -1, & 1, & -1, & 1, & -1, \ldots, \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

$$\begin{array}{cccccccc} 0, & 1, & 0, & \frac{1}{2} & 0, & \frac{1}{3}, \ldots. \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

In the first case we simply have a constant quantity: the whole "set" of the values taken by the function reduces to one value; in the second case the set contains two values taken in turn. Finally, in the third case, the set of different values taken by the function x_n is infinite while every second value of the function is zero. Thus, the domain of variation $\mathcal{X}$ of the function x_n as a variable quantity and the sequence (2) are essentially different. The first difference consists in the fact that in the set $\mathcal{X}$ every element occurs once, while in the sequence (2) one element can be repeated several (or

† Similarly we could speak of a sequence of points of a straight line or of any other object numbered by positive integral indices.

even an infinite) number of times. The second, and the most essential, difference consists in the fact that the set $\mathfrak{X}$ is "amorphous", unordered, while for the terms of the sequence (2) a definite order has been established.

The customary method of notation [see (2)] seems to imply a *spatial* location to the elements of the sequence. But such a notation is applied only for convenience and is not connected with the essence of the problem. If we say that the variable "ranges" over some sequence of values, then the reader might imagine that the variable takes its value in consecutive instances of time; in fact, however, it has nothing to do with time. Only for clarity sometimes the following expressions are used: "remote" values of the variable, starting from some "place" or from some "instant" of variation, etc.

28. Definition of the limit of a sequence. The ordering of the values of the variable x_n, according to increasing numbers which led us to consider the sequence (2) of these values, simplifies the concept of the "process" of the variable x_n approaching its limit a—as n increases to infinity.

The number a is called the *limit* of the function x_n if the latter differs from a by an arbitrarily small amount, beginning from a certain place, i.e. for all sufficiently large numbers n.

This statement clearly expresses the essence of the matter, but what "arbitrarily small" or "sufficiently great" means has to be explained. We now present a longer but comprehensive and precise definition of limit.

The number a is said to be the *limit* of the variable x_n if for any positive number ε, no matter how small it is, a number N exists such that all values of x_n the numbers of which $n > N$ satisfy the inequality

$$|x_n - a| < \varepsilon. \tag{3}$$

The fact that a is the limit of variable x_n is written as follows:

$$\lim x_n = a$$

(lim is an abbreviation of the Latin word *limes*, meaning "limit"). Sometimes it is said that the variable *tends* to a and we then write

$$x_n \to a.$$

Finally, the number a is also called the *limit of sequence* (2) and we may say that this *sequence converges to a.*

The inequality (3) where ε is arbitrary is the precise statement of the fact that x_n differs from a by "an arbitrarily small amount", and the number N indicates the "place" beginning from which this fact occurs, so that all numbers $n > N$ are "sufficiently large".

It is important to understand that in general the number N cannot be indicated once for all; it depends on the choice of the number ε. To emphasize this, instead of N we shall sometimes write N_ε. On decreasing the number ε the corresponding number $N = N_\varepsilon$ in general increases: the greater nearness of the values of x_n to a is required, the more "remote" values of it in the sequence (2) have to be considered.

An exception is provided by the case when *all* values of the variable x_n are equal to the constant number a. Obviously, then $a = \lim x_n$, but now the inequality (3) is satisfied for *an arbitrary* $\varepsilon > 0$ for *all* values of x_n†.

We know [Sec. 8] that inequality (3) is equivalent to the following:

$$-\varepsilon < x_n - a < \varepsilon$$

or

$$a - \varepsilon < x_n < a + \varepsilon; \tag{4}$$

we shall frequently make use of this fact in subsequent considerations.

The open interval $(a-\varepsilon, a+\varepsilon)$ with centre a is usually said to be a *neighbourhood* of this point. Thus, for an arbitrarily small neighbourhood of the point a all values of x_n beginning from one

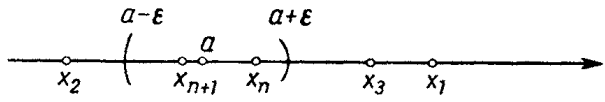

FIG. 18.

of them should be located inside the considered neighbourhood (so that outside it there remain, at most, a finite number of these values). If the number a and the values of the variable x_n are represented by points on the number axis [Sec. 13] (Fig. 18), the point

† An analogous fact occurs for the variable x_n the values of which become equal to a, beginning from some place.

representing the number a turns out to be a sort of a focus of a cluster of the points representing the values of x_n.

29. Infinitesimal quantities. The case when the variable tends to zero, $x_n \to 0$, is of a special interest.

The variable x_n the limit of which is zero is called an *infinitesimal quantity* or simply an *infinitesimal.*

If in the definition of the limit of the variable x_n [Sec. 28] we set $a = 0$, inequality (3) takes the form

$$|x_n - 0| = |x_n| < \varepsilon \quad (\text{for } n > N_\varepsilon).$$

Thus, the above given definition of the infinitesimal can be formulated at greater length without using the term "limit":

A variable x_n is called infinitesimal if for sufficiently great numbers its absolute value becomes and remains smaller than an arbitrarily small number $\varepsilon > 0$ previously prescribed.

The not too fortunate term "infinitesimal" quantity should not mislead the reader. It must be remembered that it is a variable quantity† which only in the course of its variation is capable of becoming finally smaller than an arbitrary number ε.

Returning to the general case of the variable x_n having the limit a

$$\alpha_n = x_n - a$$

we note that the difference between the variable and its limit is evidently infinitesimal, for in view of (3)

$$|\alpha_n| = |x_n - a| < \varepsilon \quad (\text{as } n > N_\varepsilon).$$

Conversely, if α_n is infinitesimal, then $x_n \to a$. This leads us to the following statement.

In order that the variable x_n should tend to the limit a, it is necessary and sufficient that the difference $\alpha_n = x_n - a$ be infinitesimal.

In this connection we could also give another definition for the concept of a "limit" (equivalent to the former one):

A constant number a is called the limit of a variable x_n if the difference between them is an infinitesimal quantity.

Obviously, if we build on this definition of the limit, we have to make use of the second of the above definitions for the infinitesimal.

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† With exception of the trivial case when it identically vanishes.

Otherwise we would be led to a vicious circle: the limit would be defined in terms of infinitesimal and the infinitesimal in terms of limit!

Thus, if the variable $x_n \to a$ it can be represented in the form

$$x_n = a + \alpha_n,$$

where α_n is infinitesimal, and conversely, if a variable admits such a representation, its limit is a. This fact is frequently used in the practical determination of the limit of a variable.

30. Examples. *(1)* Consider the variables

$$x_n = \frac{1}{n}, \quad x_n = -\frac{1}{n}, \quad x_n = \frac{(-1)^{n+1}}{n};$$

to which there correspond the following sequences of values:

$$1, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}, \ldots,$$

$$-1, \ -\frac{1}{2}, \ -\frac{1}{3}, \ -\frac{1}{4}, \ \ldots,$$

$$1, \ -\frac{1}{2}, \quad \frac{1}{3}, \ -\frac{1}{4}, \ \ldots.$$

All three variables are infinitesimals, i.e. their limits are zero. In fact, we have

$$|x_n| = \frac{1}{n} < \varepsilon,$$

if $n > 1/\varepsilon$. Thus, for N_ε we can for instance take the greatest integer contained in $1/\varepsilon$, i.e. $E(1/\varepsilon)$†.

Observe that the first variable is always greater than its limit, zero, while the second is always smaller than zero; the third now is alternatively greater and smaller than zero.

(2) If we set

$$x_n = \frac{2+(-1)^n}{n},$$

the variable ranges over the sequence of values

$$1, \ \frac{3}{2}, \ \frac{1}{3}, \ \frac{3}{4}, \ \frac{1}{5}, \ \frac{3}{6}, \ \ldots.$$

† See p. 30.

Again, in this case $x_n \to 0$, since

$$|x_n| \leqslant \frac{3}{n} < \varepsilon$$

for $n > 3/\varepsilon$ so that for N_ε we may take $E(3/\varepsilon)$.

Here we find an interesting peculiarity: the variable in turn approaches its limit, zero, and then moves away from it.

(3) Now let

$$x_n = \frac{1 + (-1)^n}{n};$$

we have already met this variable in Sec. 27. Here again $x_n \to 0$, for

$$|x_n| \leqslant \frac{2}{n} < \varepsilon,$$

if $n > N_\varepsilon = E(2/\varepsilon)$.

Observe that, for all odd values of n, the variable is equal to its limit.

These simple examples are interesting since they describe the variety of possibilities which the above given definition of the limit contains. It is irrelevant whether these values of the variable are located on one side of the limit or not; it is irrelevant whether the variable approaches with every step its limit; finally it is irrelevant whether it reaches the limit, i.e. whether it takes the values equal to the limit. As stated in the definition, it is only essential that the variable should finally, i.e. for sufficiently large values of the independent variable, differ from its limit by an arbitrarily small amount.

(4) Define the variable by the formula

$$x_n = a^{1/n} = \sqrt[n]{a} \quad (a > 1);$$

we shall prove that $x_n \to 1$.

If we make use of relation (3) of Sec. 11 we may write

$$|x_n - 1| = \sqrt[n]{a} - 1 < \frac{a-1}{n} < \varepsilon, \quad \text{if only} \quad n > N_\varepsilon = E\left(\frac{a-1}{\varepsilon}\right).$$

However, we can also reason in a different way. The inequality

$$|x_n - 1| = a^{1/n} - 1 < \varepsilon$$

is equivalent to the following:

$$\frac{1}{n} < \log_a(1+\varepsilon) \quad \text{or} \quad n > \frac{1}{\log_a(1+\varepsilon)},$$

hence it is satisfied for $n > N_\varepsilon = E[1/\log_a(1+\varepsilon)]$.

Following those two ways of reasoning we have arrived at distinct expressions for N_ε. For instance, for $a = 10$, $\varepsilon = 0.01$ we obtain $N_{0.01} = 9/0.01 = 900$ according to the first method, and $N_{0.01} = E(1/0.00432) = 231$ according to

the second method. Using the second method we obtained the smallest possible value for $N_{0.01}$, for $10^{1/231} = 1.010017...$ which differs from unity by more than $\varepsilon = 0.01$. This is also the case for any ε, a.

We note that we are not at all interested in the smallest possible value of N_ε, if we only want to establish the fact of "tending to a limit"; also, the inequality (3) should be satisfied, beginning from any point, however large or small.

(5) An important example of an infinitesimal is provided by the variable

$$\alpha_n = q^n \quad \text{where} \quad |q| < 1.$$

To prove that $\alpha_n \to 0$ consider the inequality

$$|\alpha_n| = |q|^n < \varepsilon;$$

it is equivalent to the following:

$$n \cdot \log|q| < \log \varepsilon \quad \text{or} \quad n > \frac{\log \varepsilon}{\log|q|}\dagger.$$

Thus, if we set (assuming $\varepsilon < 1$)

$$N_\varepsilon = E\left(\frac{\log \varepsilon}{\log|q|}\right),$$

then for $n > N_\varepsilon$ the above inequality is certainly satisfied.

Similarly, it is easy to establish also that the variable

$$\beta_n = Aq^n,$$

where as before $|q| < 1$ and A is a constant number, is also an infinitesimal quantity.

(6) Lastly, we consider the infinite decreasing geometric progression

$$\therefore a, aq, aq^2, \ldots, aq^{n-1}, \ldots \quad (|q| < 1)$$

and we proceed to find its sum.

It is known that by the sum of infinite progression we understand the limit to which the sum s_n of n terms of the progression tends, when n tends to infinity. But

$$s_n = \frac{a - aq^n}{1-q} = \frac{a}{1-q} - \frac{a}{1-q} \cdot q^n,$$

so that the variable s_n differs from $a/(1-q)$ by $-aq^n/(1-q)$ which, as we have just found, is infinitesimal. Consequently, according to the second definition of the limit, the required sum of the progression

$$s = \lim s_n = \frac{a}{1-q}.$$

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† It should be borne in mind that $|q| < 1$ and $\log|q| < 0$; hence, in dividing both sides of the inequality by this number the inequality sign should be reversed.

31. Infinitely large quantities. The opposite of infinitely small quantities are, in a way, infinitely large quantities.

The variable x_n is called infinitely large, if for sufficiently large values of n its absolute value becomes and remains greater than arbitrarily large prescribed number $\mathrm{E} > 0$,

$$|x_n| > \mathrm{E} \quad (\text{for } n > N_{\mathrm{E}}).$$

As in the case of infinitesimals we emphasize that no isolated value of an infinitely large quantity can be regarded as "large"; we have a variable quantity which, only in the course of variation, will finally become greater than an arbitrary number E.

The following are examples of infinitely large quantities:

$$x_n = n, \quad x_n = -n, \quad x_n = (-1)^{n+1} n;$$

they range over the set of positive integers, the first with positive sign, the second with negative and the third with alternating sign.

Another example of an infinitely large quantity is

$$x_n = Q^n, \quad \text{for} \quad |Q| > 1.$$

In fact, for any $E > 0$, the inequality

$$|x_n| = |Q|^n > \mathrm{E}$$

is valid, provided that

$$n \cdot \log |Q| > \log \mathrm{E} \quad \text{or} \quad n > \frac{\log \mathrm{E}}{\log |Q|}\dagger.$$

hence for N_{E} we can take the number

$$E\left(\frac{\log \mathrm{E}}{\log |Q|}\right).$$

Particularly important are the cases when the infinitely large quantity x_n (at least for sufficiently large n) has a constant sign ($+$ or $-$); then, in accordance with the sign, it is said that the variable x_n has the limit $+\infty$ or $-\infty$, and also that it tends to $+\infty$ or $-\infty$; we then write

$$\lim x_n = +\infty, \; x_n \to +\infty \text{ or } \lim x_n = -\infty, \; x_n \to -\infty.$$

We could formulate for these cases an independent definition, replacing the inequality $|x_n| > \mathrm{E}$, according to the case considered, by the inequality

$$x_n > \mathrm{E} \quad \text{or} \quad x_n < -\mathrm{E},$$

which already implies that $x_n > 0$ or $x_n < 0$, respectively.

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† Since $|Q| > 1$, $\log |Q| > 0$.

It is evident, in the general case, that the infinitely large quantity x_n is characterized by the relation $|x_n| \to +\infty$.

It is evident from the above examples of infinitely large quantities that the variable $x_n = n$ tends to $+\infty$, the variable $x_n = -n$ to $-\infty$. Now, we cannot say of the third variable $x_n = (-1)^{n+1}n$ that it tends either to $+\infty$ or to $-\infty$.

Finally, for the variable $x_n = Q^n$, only for $Q > 1$ may we say that it tends to $+\infty$; when $Q < -1$ it has no limit.

We have already encountered the "improper numbers" $\pm\infty$ in Sec. 6; it should be borne in mind that their application is of conditional nature and we should be careful not to perform upon these numbers any arithmetical operations. Instead of $+\infty$ one frequently writes simply ∞.

To conclude we mention a simple connection existing between infinitely large and infinitely small quantities.

If a variable x_n is infinitely large, then the inverse quantity $\alpha_n = 1/x_n$ is infinitely small.

Take any number $\varepsilon > 0$. By definition of the infinitely large quantity, for the number $E = 1/\varepsilon$ a number N can be found such that

$$|x_n| > \frac{1}{\varepsilon}, \quad \text{if only} \quad n > N.$$

Evidently, for the same n we have

$$|\alpha_n| < \varepsilon,$$

which proves our statement.

Similarly the converse statement can be proved:

If a variable α_n (non-vanishing) is infinitely small, then the inverse quantity $x_n = 1/\alpha_n$ is infinitely large.

32. Definition of the limit of a function. Consider the number set $\mathcal{X} = \{x\}$. The point a is called the *point of condensation*† of this set if, in an arbitrary neighbourhood $(a-\delta, a+\delta)$ [Sec. 28] of this point, there are values of x from $\mathcal{X}$ distinct from a. The actual point of condensation may or may not belong to the set $\mathcal{X}$. For instance, if $\mathcal{X} = [a, b]$ or $\mathcal{X} = (a, b]$ then in both cases a is the point of condensation for $\mathcal{X}$ but in the first case it belongs to $\mathcal{X}$ while in the second it does not.

† Or, *point of accumulation* [Ed.].

Assuming that a is the point of condensation for $\mathcal{X}$ we can extract from $\mathcal{X}$, in an infinite number of ways, the sequence

$$x_1, x_2, x_3, \ldots, x_n, \ldots \tag{2}$$

of values of x distinct from a, the limit of which is a. In fact, prescribing a sequence of positive numbers δ_n converging to zero, in every neighbourhood $(a-\delta_n, a+\delta_n)$ of the point a (for $n = 1, 2, 3, \ldots$), we find a point $x = x_n$ from $\mathcal{X}$ distinct from a; since $\delta_n \to 0$ and $|x_n - a| < \delta_n$ we have that $x_n \to a$.

Consider now a function $f(x)$ defined over the domain $\mathcal{X}$ for which a is a point of condensation. It is of interest to investigate the behaviour of this function when x tends to a. It is said that the function $f(x)$ has the limit A, finite or otherwise, when x tends to a (or briefly at the point a), if for any sequence (2) of the variable x extracted from $\mathcal{X}$, with a limit a, the corresponding sequence of values of the function

$$f(x_1), f(x_2), f(x_3), \ldots, f(x_n), \ldots \tag{5}$$

always has the limit A. This is written as follows:

$$\lim_{x \to a} f(x) = A \tag{6}$$

or

$$f(x) \to A \quad \text{for} \quad x \to a. \tag{7}$$

Suppose now that the set $\mathcal{X} = \{x\}$ contains arbitrarily large positive values of x; then it is said that $+\infty$ is the point of condensation of this set. If by the neighbourhood of the point $+\infty$ we understand the interval $(\Delta, +\infty)$, then the above statement can have the following form: numbers of the set $\mathcal{X}$ should be contained in every neighbourhood of the point $+\infty$.

If this is satisfied we can extract from $\mathcal{X}$ the sequence (2) having the limit $+\infty$. In fact, taking an arbitrary positive variable Δ_n tending to $+\infty$, for any $\Delta_n (n = 1, 2, 3, \ldots)$ we find in $\mathcal{X}$ a value $x_n > \Delta_n$; evidently, $x_n \to +\infty$.

Assuming that $+\infty$ is a point of condensation for $\mathcal{X}$, consider a function $f(x)$ defined over this domain. For this function we can establish the concept of a limit as $x \to +\infty$

$$\lim_{x \to +\infty} f(x) = A$$

exactly as it was done before—simply replacing a by $+\infty$.

Similarly, we establish the concept of the limit of the function $f(x)$ when $x \to -\infty$:

$$\lim_{x \to -\infty} f(x) = A.$$

Here we have to assume beforehand that $-\infty$ is a point of condensation of the set $\mathfrak{X}$; the meaning of this statement is clear.

To conclude we consider an extension to the general case of the limit of a function and of the terminology established in Secs. 29 and 31 for a function of positive integral argument. Suppose that in a definite passage to the limit of x, the function $f(x)$ tends to zero; then this function is called an infinitely small quantity. If the function $f(x)$ tends to a finite limit A then the difference $f(x)-A$ is infinitesimal, and conversely. When $|f(x)|$ tends to $+\infty$ it is said to be an infinitely large quantity†. Finally, it is easy to extend to the considered general case the theorems at the end of Sec. 31, establishing the relation between infinitely small and infinitely large quantities.

33. Another definition of the limit of a function. The concept of a limit of a function $f(x)$ when x tends to a has been constructed on the basis of the more fundamental concept of limit of a sequence examined earlier. However, we can present another definition of limit of a function, without using at all the concept of a limit of a sequence.

We first confine ourselves to the case when both numbers a and A are finite. Then, assuming that a is a point of condensation of the domain $\mathfrak{X}$ where the function $f(x)$ is given, the new definition of the limit is as follows:

A function $f(x)$ has the limit A when x tends to a, if for any number $\varepsilon > 0$ a number $\delta > 0$ can be found such that

$$|f(x) - A| < \varepsilon, \quad \text{if only} \quad |x - a| < \delta \tag{8}$$

(x being taken from $\mathfrak{X}$ and is distinct from a)‡.

This definition is entirely equivalent to that given in Sec. 32. To prove this assertion, we assume first that the condition just

† If this occurs when $x \to a$ where a is finite, it is also said that at the point a the function is infinite.

‡ From the fact that a is a point of condensation for $\mathfrak{X}$ it follows that such values of x certainly exist in the neighbourhood $(a-\delta, a+\delta)$ of a.

formulated is satisfied, and according to an arbitrary $\varepsilon > 0$ the corresponding (in the stated meaning) number $\delta > 0$ has been found. Let us extract from $\mathfrak{X}$ an arbitrary sequence (2) converging to a, (all x_n are distinct from a). By definition of the limit of the sequence, to the number $\delta > 0$ there corresponds a number N such that for $n > N$ the inequality $|x_n - a| < \delta$ is satisfied, and consequently (see (8)) $|f(x_n) - A| < \varepsilon$, also. This proves the convergence of the sequence (5) to A. Thus, the condition of the earlier definition is satisfied.

Assume now that the limit of the function exists according to the earlier definition. To prove that then the condition of the new definition is also satisfied, assume the converse. Then for some number $\varepsilon > 0$ the corresponding number δ would not exist, i.e. no matter how small δ we take, always at least one value of the variable $x = x'$ can be found (distinct from a) for which

$$|x' - a| < \delta, \quad \text{none the less} \quad |f(x') - A| \geqslant \varepsilon.$$

Take a sequence of positive numbers δ_n converging to zero. On the basis of what we said above, for every number $\delta = \delta_n$ a value $x' = x'_n$ can be found, such that

$$|x'_n - a| < \delta_n; \quad \text{none the less,} \quad |f(x'_n) - A| \geqslant \varepsilon.$$

Thus, from these values we can construct the sequence

$$x'_1, x'_2, x'_3, \ldots, x'_n \ldots,$$

for which

$$|x'_n - a| < \delta_n \qquad (n = 1, 2, 3, \ldots);$$

since $\delta_n \to 0$, we see that $x'_n \to a$.

By hypothesis, the corresponding sequence of the values of the function

$$f(x'_1), f(x'_2), f(x'_3), \ldots, f(x'_n), \ldots$$

should converge to A, but this is impossible owing to the fact that for all $n = 1, 2, 3, \ldots$ we have $|f(x'_n) - A| \geqslant \varepsilon$. This contradiction proves the assertion.

We can easily formulate the new definition of the limit for the cases when one, or both, of the numbers a, A are equal to $+\infty$ or $-\infty$. We give, for example, the full statement of the definition for the case $a = +\infty$ and A finite (or also equal to $+\infty$).

The function $f(x)$ has the finite limit A when x tends to $+\infty$, if for any number $\varepsilon > 0$ $(E > 0)$ a number $\Delta > 0$ can be found, such that

$$|f(x) - A| < \varepsilon (f(x) > E), \quad \text{if only} \quad x > \Delta \quad (x \text{ in } \mathfrak{X}).$$

The proof of equivalency of this definition to the definition "in the language of sequences" is the same as before.

If we apply this definition to the variable x_n as a function of the independent variable n, for $n \to +\infty$, we return to the original definition of the limit of such a function, or, equivalently, to the limit of the sequence; this definition was given in Secs. 28 and 31 (the role of the number Δ was there played by N). Thus, the former definition of the limit of a function reduced this concept to limit of a sequence, while the definition of the limit of a sequence turns out to be simply a particular case of the definition of the limit of a function in general, when the new form is used. The limit which was before denoted by

$$\lim x_n$$

should now be denoted by

$$\lim_{n \to +\infty} x_n.$$

Incidentally, in fact the index $n \to +\infty$ can always be omitted without causing a misunderstanding, since no other passage to the limit can be meant; the domain $\mathfrak{N}$ of variation of the positive integer n has the *unique* point of condensation $+\infty$.

In spite of the difference in definitions of the limit of the function (in the new form) as applied to various assumptions with respect to a and A, the essence is the same, namely the function should be contained in an arbitrary "neighbourhood" of its limit A, provided that the independent variable is contained in the appropriately selected "neighbourhood" of its limit a.

Thus, for the concept of the limit of function we have two equivalent definitions; we shall, in any given case, use the one which is more convenient.

34. Examples. *(1)* As in the proof of Sec. 30, *(5)* of the limit relation

$$\lim a^{1/n} = 1 \quad (a > 1)$$

we can obtain a more general one,

$$\lim_{x \to 0} a^x = 1 \quad (a > 1).$$

It is required to find for a given $\varepsilon > 0$† a $\delta > 0$ such that

$$|a^x - 1| < \varepsilon, \quad \text{provided that} \quad |x| < \delta.$$

But the first inequality or the equivalent inequalities

$$1 - \varepsilon < a^x < 1 + \varepsilon$$

are satisfied, if

$$\log_a(1 - \varepsilon) < x < \log_a(1 + \varepsilon).$$

Since

$$\log_a(1 - \varepsilon) + \log_a(1 + \varepsilon) = \log_a(1 - \varepsilon^2) < 0 \text{ and } \log_a(1 - \varepsilon) < -\log_a(1 + \varepsilon),$$

the above mentioned inequalities are certainly satisfied, if

$$-\log_a(1 + \varepsilon) < x < \log_a(1 + \varepsilon) \quad \text{or} \quad |x| < \log_a(1 + \varepsilon).$$

Thus, it is sufficient to set $\delta = \log_a(1 + \varepsilon)$ in order that for $|x| < \delta |a^x - 1| < \varepsilon$. This completes the proof.

(2) We now prove that

$$\lim_{x \to +\infty} a^x = +\infty \quad (\text{for } a > 1).$$

For an arbitrary $E > 0$, it is sufficient to take $\Delta = \log_a E$ in order that

$$x > \Delta \quad \text{implies that } a^x > E,$$

which proves our assertion‡.

Similarly we can prove that

$$\lim_{x \to -\infty} a^x = 0 \quad (\text{for } a > 1).$$

In fact, for any $\varepsilon > 0$ $(\varepsilon < 1)$, if we take

$$\Delta = \log_a \frac{1}{\varepsilon} = -\log_a \varepsilon,$$

then as $x < -\Delta$ we necessarily have $a^x < \varepsilon$.

If now $0 < a < 1$, by means of the transformation

$$a^x = \left(\frac{1}{a}\right)^{-x}$$

it is easy to establish the result

$$\lim_{x \to +\infty} a^x = 0, \quad \lim_{x \to -\infty} a^x = +\infty, \quad (\text{for } 0 < a < 1).$$

---

† There is no reason why we should not take $\varepsilon < 1$.

‡ The particular case

$$\lim a^n = +\infty$$

was already dealt with in Sec. 31.

(3) Let us prove that for $a>1$ and $x>0$

$$\lim_{x\to+\infty}\log_a x=+\infty,\quad \lim_{x\to 0}\log_a x=-\infty.$$

For an arbitrary $E>0$, provided that $x>a^E$, we have $\log_a x>E$ and, similarly, if $0<x<a^{-E}$ the inequality $\log_a x<-E$ is satisfied. This proves the two relations.

(4) Further, we have

$$\lim_{x\to+\infty}\operatorname{arc\,tan} x=\frac{\pi}{2},\quad \lim_{x\to-\infty}\operatorname{arc\,tan} x=-\frac{\pi}{2}.$$

Let us for instance examine the first limit. For any $\varepsilon>0$ it is sufficient to take $x>\tan[(\pi/2)-\varepsilon]$ in order that $\operatorname{arc\,tan} x>(\pi/2)-\varepsilon$; hence

$$0<\frac{\pi}{2}-\operatorname{arc\,tan} x<\varepsilon.$$

(5) We now establish the following result:

$$\lim_{x\to 0}\frac{\sin x}{x}=1. \tag{9}$$

However, we have first to prove the useful inequalities

$$\sin x<x<\tan x \quad \text{for} \quad 0<x<\frac{\pi}{2}. \tag{10}$$

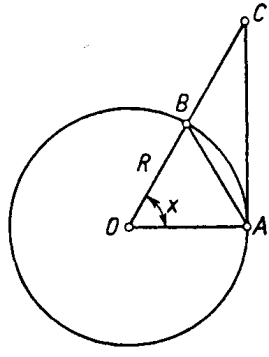

FIG. 19.

For this purpose, consider in the circle of radius R, the acute angle AOB, the chord AB and the tangent AC to the circle at the point A (see Fig. 19). Then we have: the area $\triangle AOB<$ the area of the sector $AOB<$ the area $\triangle AOC$†.

† We make use here of the knowledge of areas of elementary figures, treated in any school course.

If by x we denote the measure in radians of the angle AOB, the length of the arc AB is given by the product Rx and the inequalities take the form

$$\tfrac{1}{2}R^2 \sin x < \tfrac{1}{2} R^2 x < \tfrac{1}{2}R^2 \tan x.$$

Hence, dividing by $R^2/2$, we arrive at inequalities (10).

Assuming that $0 < x < \pi/2$ let us divide $\sin x$ by each term of inequality (10). Then we obtain

$$1 > \frac{\sin x}{x} > \cos x,$$

whence

$$0 < 1 - \frac{\sin x}{x} < 1 - \cos x.$$

Now

$$1 - \cos x = 2\sin^2 \frac{x}{2} < 2\sin \frac{x}{2} < x$$

(in view of (10)) and hence

$$0 < 1 - \frac{\sin x}{x} < x.$$

This implies the inequality

$$\left| \frac{\sin x}{x} - 1 \right| < |x|,$$

which, obviously, holds also if the sign of x is changed, i.e. it is valid for all $x \neq 0$, provided that $|x| < \pi/2$.

This inequality solves the problem. In fact, if the number $\varepsilon > 0$ is arbitrarily taken, then for δ it is sufficient to take the smallest of the numbers $\varepsilon, \pi/2$: for $|x| < \delta$ the inequality holds (for $\delta \leqslant \pi/2$), and from it (since $\delta \leqslant \varepsilon$) it follows that

$$\left| \frac{\sin x}{x} - 1 \right| < \varepsilon.$$

(6) Finally, it is interesting to examine a case when the limit of a function does *not* exist: the function $\sin x$ when x tends to $+\infty$ $(-\infty)$ has no limit at all.

To prove the absence of the limit we may simply assume "the standpoint of sequences". It is sufficient to observe that to the two sequences

$$\left\{\frac{2n-1}{2}\pi\right\} \quad \text{and} \quad \left\{\frac{2n+1}{2}\pi\right\} \quad (n = 1, 2, 3, \ldots)$$

of values x having the limit $+\infty$, there correspond the sequences of values of the function tending to distinct limits

$$\sin\frac{2n-1}{2}\pi = -1 \to -1, \quad \sin\frac{2n+1}{2}\pi = 1 \to 1.$$

If we remember the "oscillatory" nature of the sinusoid, the absence of the limit for this case becomes clear.

Similarly, the function $\sin(1/\alpha)$ when α tends to zero (both for $\alpha > 0$ and $\alpha < 0$) has no limit. In essence, this is only another form of the above example; replacing in the function $\sin x$, x by $1/\alpha$, we obtain the new form. It is evident that when α ranges over the sequence of positive (negative) values approaching zero, then $x = 1/\alpha$ tends to $+\infty$ $(-\infty)$, and conversely.

Let us again write in the expression $\sin(1/\alpha)$ instead of α the letter x (in order to return to the customary notation for the abscissa) and consider the graph of the function

$$y = \sin\frac{1}{x} \quad (x \neq 0),$$

confining ourselves to the values of x from 0 to $2/\pi$ (and from $-2/\pi$ to 0).

Note that the values of x decreasing steadily to zero:

$$\frac{2}{\pi}, \frac{1}{\pi}, \frac{2}{3\pi}, \frac{1}{2\pi}, \frac{2}{5\pi}, \frac{1}{3\pi}, \frac{2}{7\pi}, \ldots, \frac{2}{(2n-1)\pi}, \frac{1}{n\pi}, \frac{2}{(2n+1)\pi}, \ldots$$

correspond to the values of $1/x$ increasing to ∞:

$$\frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \frac{5\pi}{2}, 3\pi, \frac{7\pi}{2}, \ldots, \frac{(2n-1)\pi}{2}, n\pi, \frac{(2n+1)\pi}{2}, \ldots.$$

In the intervals between the above values (for decreasing x) our function alternately decreases from 1 to 0 and from 0 to -1, then increases from -1 to 0 and from 0 to 1, etc. Thus, the function $\sin(1/x)$ performs an infinite number of oscillations, just as does the function $\sin x$, but in the case of the latter function these oscillations are distributed over an infinite interval, while in our case they are all contained in a finite interval, condensing to zero.

The graph is given in Fig. 20 (of course not wholly, since an infinite number of oscillations cannot be reproduced). Since on changing the sign of x, $\sin(1/x)$ also changes its sign, the left-hand part of the graph is symmetric to the right-hand part about the origin.

(7) If for $x \neq 0$ we consider the function $x\sin(1/x)$ which differs by the factor x from the function $\sin(1/x)$ just examined, we observe that now the limit when $x \to 0$ does exist

$$\lim_{x\to 0} x \cdot \sin\frac{1}{x} = 0;$$

this is clear from the inequality

$$\left|x \cdot \sin\frac{1}{x}\right| \leqslant |x|.$$

When x approaches zero, our function as before performs an infinite number of oscillations, but their amplitude (owing to the presence of the factor x) decreases, tends to zero; this fact ensures the existence of the limit.

The graph of the function

$$y = x \cdot \sin\frac{1}{x}$$

is shown in Fig. 21; it is contained between the two bisectors $y = x$ and $y = -x$ of the first and third, and second and fourth quadrants†.

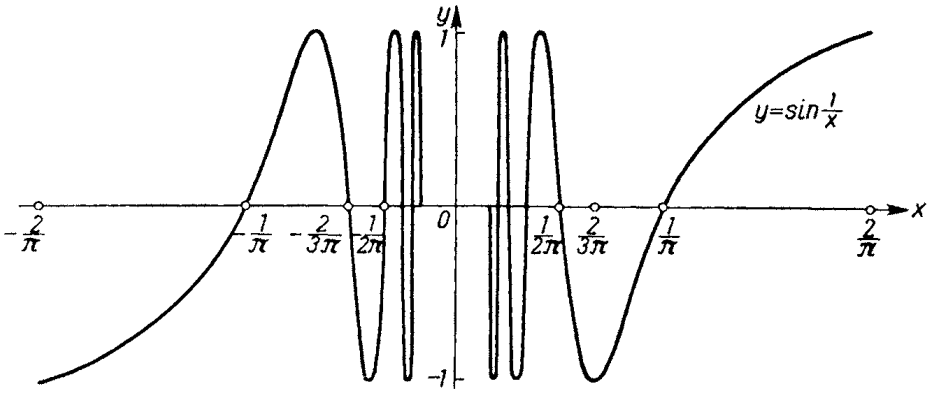

FIG. 20.

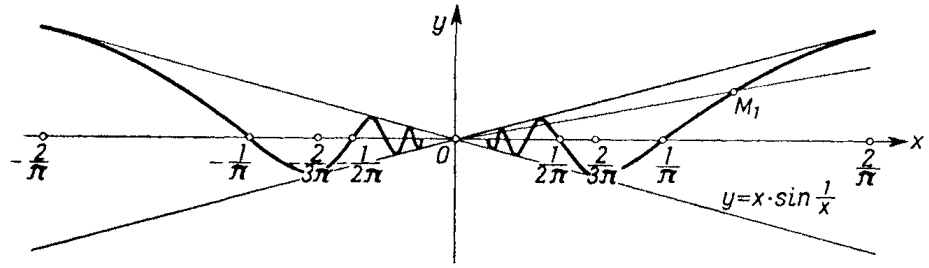

FIG. 21.

Remark. We have the limits

$$\lim_{x\to 0}\frac{\sin x}{x} = 1, \quad \lim_{x\to 0} x \cdot \sin\frac{1}{x} = 0,$$

which have the common characteristic that neither of these functions is defined for $x = 0$. This does not prevent us from speaking about their limits as $x \to 0$,

† In Figs. 20 and 21 for clarity we had to take a greater scale on the x-axis, which leads to a distortion.

since according to the precise meaning of our definition, the value $x = 0$ is not considered at all.

Similarly, the fact that the function $\sin(1/x)$ has no meaning when $x = 0$ does not prevent us from considering the question of the existence of a limit as $x \to 0$; here, however, it turns out that such a limit does not exist.

35. One-sided limits. If the domain $\mathfrak{X}$ is such that values of x from $\mathfrak{X}$ can be found arbitrarily close to a on the right-hand side, then the definition of the limit presented in Secs. 32 and 33 can be particularized to the values $x > a$. In this case the limit of the function, if it exists, is called the limit of the function $f(x)$ when x tends to a from the right (or briefly, at the point a from the right) and it is denoted by the symbol

$$\lim_{x \to a+0} f(x) \quad \text{or} \quad f(a+0).$$

Similarly we define the concept of the limit of function when x tends to a (at the point a) from the left

$$\lim_{x \to a-0} f(x) \quad \text{or} \quad f(a-0)^{\dagger}.$$

These two limits are called *one-sided.*

If the domain $\mathfrak{X}$ is such that tending to a is possible both from left and from right, then we may consider both limits. It can easily be established that for the existence of the ordinary (two-sided) limit (6) it is necessary and sufficient that the two one-sided limits exist and that they are equal:

$$\lim_{x \to a+0} f(x) = \lim_{x \to a-0} f(x) = A.$$

Observe that these limits may exist and be unequal. Examples can easily be constructed, on the basis of Examples *(1)* and *(4)* examined in Sec. 34.

Examples. Define two functions for $x \neq 0$ by the relations

$$f_1(x) = a^{\frac{1}{x}} \quad (a > 1), \quad f_2(x) = \arctan \frac{1}{x}.$$

For the first we have

$$f_1(+0) = \lim_{x \to +0} a^{\frac{1}{x}} = \lim_{z \to +\infty} a^z = +\infty,$$

$$f_1(-0) = \lim_{x \to -0} a^{\frac{1}{x}} = \lim_{z \to -\infty} a^z = 0,$$

† If $a = 0$, instead of $0+0\,(0-0)$ we simply write $+0\,(-0)$.

while for the second

$$f_2(+0) = \lim_{x \to +0} \operatorname{arc\,tan} \frac{1}{x} = \lim_{z \to +\infty} \operatorname{arc\,tan} z = \frac{\pi}{2},$$

$$f_2(-0) = \lim_{x \to -0} \operatorname{arc\,tan} \frac{1}{x} = \lim_{z \to -\infty} \operatorname{arc\,tan} z = -\frac{\pi}{2}.$$

The graphs of these functions are given in Figs. 22 and 23.

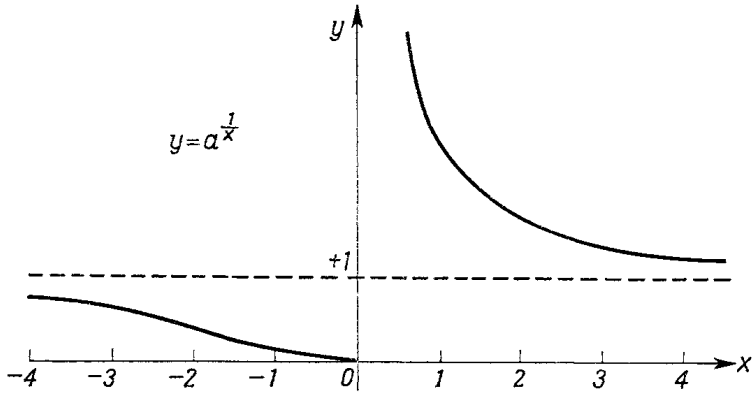

FIG. 22.

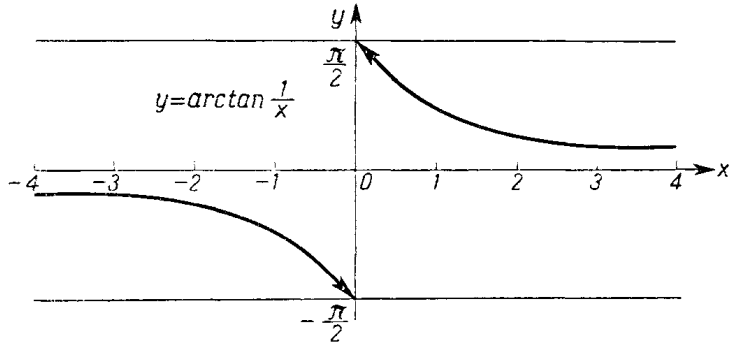

FIG. 23.

§ 2. Theorems on limits

36. Properties of functions of a positive integral argument, possessing a finite limit. Since the formulation and proof of theorems concerning a function of positive integral argument are simpler than those for a general function, we shall always first state and prove theorems for this particular case and only then shall we remark on the extension to the general case.

(1) If the variable x_n tends to a limit a and $a > p$ $(a < q)$, then all values of the variable beginning from a certain one are also greater than p (smaller than q).

Selecting a positive number $\varepsilon < a - p$ $(q - a)$ we have

$$a - \varepsilon > p \quad (a + \varepsilon < q).$$

But, by the definition of the limit of the variable x_n [Sec. 28] for this ε a number N can be found such that for $n > N$ we have

$$a - \varepsilon < x_n < a + \varepsilon.$$

For these values of n we certainly have $x_n > p$ $(x_n < q)$.

This simple proposition has a number of important corollaries.

(2) If the variable x_n tends to a limit $a > 0$ (< 0), then the variable itself $x_n > 0$ (< 0) for sufficiently large n.

To prove the statement it is sufficient to apply the preceding assertion, taking $p = 0$ $(q = 0)$.

(3) If the variable x_n tends to a limit a and, for all n,

$$x_n \leqslant p \quad (\geqslant q),$$

then also

$$a \leqslant p \quad (\geqslant q).$$

The proof is carried out by assuming the converse and using *(1)*.

From *(1)* we can now prove the *uniqueness* of the limit.

(4) The variable x_n cannot tend simultaneously to two distinct (finite) limits.

In fact, assume the converse: let, at the same time, $x_n \to a$ and $x_n \to b$ where $a < b$. Take any number r between a and b

$$a < r < b.$$

Since $x_n \to a$ and $a < r$, a number N' can be found such that for $n > N'$ the inequality $x_n < r$ holds. On the other hand, if $x_n \to b$ and $b > r$ a number N'' can be found, such that for $n > N''$ we have $x_n > r$. If the number n be taken greater than N' and N'' then the corresponding value of the variable x_n is at the same time smaller than r and greater than r, which is impossible.

This contradiction proves our assertion.

(5) If the variable x_n has a finite limit then it is bounded in the sense that all its values are contained between two finite limits

$$m \leqslant x_n \leqslant M \quad (n = 1, 2, 3, \ldots). \tag{1}$$

First, it is clear directly from the definition of the limit that for any $\varepsilon > 0$ a number N can be found such that for $n > N$ we have

$$a - \varepsilon < x_n < a + \varepsilon.$$

Thus, for $n = N+1, N+2, \ldots$, the values of x_n lie between the bounds $a-\varepsilon$ and $a+\varepsilon$. Outside these bounds there can lie some of the first N values

$$x_1, x_2, \ldots, x_N.$$

Since there is only a finite number of such exceptional values the above bounds can be widened in such a way that all values of x_n are contained inside the new bounds m and M. For instance, we can take for m the smallest of the numbers

$$a - \varepsilon, x_1, x_2, \ldots, x_N,$$

and for M the largest of the numbers

$$a + \varepsilon, x_1, x_2, \ldots, x_N.$$

Remark. In particular, it is now obvious that a variable having a finite limit cannot at the same time tend either to $+\infty$ or to $-\infty$. This is an appendix to *(4)* on the uniqueness of the limit.

37. Extension to the case of a function of an arbitrary variable. It is easy to rephrase the contents of Sec. 36 for the general case of a function $f(x)$ given in a domain $\mathfrak{X}$ with the condensation point a†.

(1) If, when x tends to a, the function $f(x)$ has a finite limit A and $A > p$ $(A < q)$, then for values of x sufficiently near a (but distinct from a) the function also satisfies the inequality

$$f(x) > p \quad (f(x) < q). \tag{2}$$

Selecting a positive number $\varepsilon < A - p$ $(q - A)$ we have

$$A - \varepsilon > p \quad (A + \varepsilon < q).$$

But, according to the second definition of the limit of a function [Sec. 33], for this ε a number δ can be found such that provided $|x-a| < \delta$ (x being taken from $\mathfrak{X}$ and distinct from a) we have

$$A - \varepsilon < f(x) < A + \varepsilon.$$

For these values of x (2) is clearly satisfied.

† The number a can be $-\infty$ or $+\infty$ but for definiteness we confine ourselves to the case of a finite a.

The reader should note that no new ideas have to be employed in this proof.

Hence we can directly prove the assertions analogous to *(2)*, *(3)* and *(4)* of Sec. 36. For instance, setting in *(1)* $p = 0$ $(q = 0)$ we obtain:

(2) If for $x \to a$ the function $f(x)$ has a finite positive (negative) limit, then the function itself is positive (negative), at least for values of x sufficiently close to a but distinct from a.

Also the assertion analogous to *(5)* is true, but in a weaker form:

(3) If when x tends to a the function $f(x)$ has a finite limit A, then for the values of x sufficiently close to a, the function is bounded in the sense that its values are contained between two finite bounds

$$m \leqslant f(x) \leqslant M \quad \text{only if} \quad 0 < |x - a| < \delta.$$

In fact, according to the definition of the limit, given $\varepsilon > 0$ we find $\delta > 0$ such that

$$A - \varepsilon < f(x) < A + \varepsilon, \quad \text{if} \quad 0 < |x - a| < \delta.$$

We recall that a similar result was originally derived also for the variable x_n; the inequalities

$$a - \varepsilon < x_n < a + \varepsilon$$

were satisfied only for $n > N$. But, in the former case, outside these bounds only a finite number of values could be found and it was easy to find the new bounds between which all values of x_n would be contained. However, in general, now we cannot do so, since there can be an infinite number of the values of x for which $|x-a| \geqslant \delta$. For instance, the function $f(x) = 1/x$ (for $x > 0$) when $x \to 1$ tends to unity; evidently $0 < f(x) < 2$ if $|x - 1| < 1/2$ but for all considered values of x the function $f(x)$ is not bounded: when $x \to +0$ it tends to $+\infty$.

38. Passage to the limit in equalities and inequalities. When connecting two variables x_n and y_n in an equality or inequality, we always understand their corresponding values, namely the values with the same number n.

(1) If two variables x_n, y_n are equal for all their variations, i.e. $x_n = y_n$, and both have finite limits

$$\lim x_n = a, \quad \lim y_n = b,$$

then the limits are also equal, i.e. $a = b$.

This assertion follows directly from the uniqueness of the limit [Sec. 36, *(4)*].

This result is usually written in the form of the limit passage in equality: if $x_n = y_n$ it is inferred that $\lim x_n = \lim y_n$.

(2) If for two variables x_n, y_n we have $x_n \geqslant y_n$ for all n, and both have finite limits

$$\lim x_n = a, \quad \lim y_n = b,$$

then $a \geqslant b$.

Assume the converse: let $a < b$. We reason in a manner similar to that of Sec. 36, *(4)*: take a number r between a and b so that $a < r < b$. Then a number N' can be found such that for $n > N'$ we have $x_n < r$; on the other hand a number N'' can be found such that for $n > N''$ we obtain $y_n > r$. If N is greater than both numbers N', N'', then for $n > N$ the two inequalities

$$x_n < r, \quad y_n > r, \quad \text{whence} \quad x_n < y_n,$$

are satisfied simultaneously. This contradicts the assumption and completes the proof of the theorem.

This theorem establishes the legitimacy of passing to the limit in $x_n \geqslant y_n$; i.e., from this we may infer that $\lim x_n \geqslant \lim y_n$.

Of course, the sign $\geqslant$ can always be replaced by the sign $\leqslant$.

We draw the reader's attention to the fact that in general $x_n > y_n$ does not imply that $\lim x_n > \lim y_n$ but only, as before, that $\lim x_n \geqslant \lim y_n$. For instance, $1/n > -1/n$ for all n; nevertheless

$$\lim \frac{1}{n} = \lim\left(-\frac{1}{n}\right) = 0.$$

We can derive the assertion *(3)* of Sec. 36 from the result *(2)* as a particular case.

In establishing the existence and value of the limit it is frequently useful to use the following result.

(3) If for the variables x_n, y_n, z_n we always have the inequalities

$$x_n \leqslant y_n \leqslant z_n,$$

and if the variables x_n and z_n tend to the common limit a

$$\lim x_n = \lim z_n = a,$$

then the variable y_n also has the same limit, i.e.

$$\lim y_n = a.$$

Take an arbitrary $\varepsilon > 0$. First, for this ε a number N' can be found such that for $n > N'$

$$a - \varepsilon < x_n < a + \varepsilon.$$

Next, a number N'' can be found such that for $n > N''$

$$a - \varepsilon < z_n < a + \varepsilon.$$

Let N be greater than both numbers N' and N''. Then for $n > N$ both of the above double inequalities are satisfied and hence

$$a - \varepsilon < x_n \leqslant y_n \leqslant z_n < a + \varepsilon.$$

Finally, for $n > N$

$$a - \varepsilon < y_n < a + \varepsilon \quad \text{or} \quad |y_n - a| < \varepsilon.$$

Thus, in fact, $\lim y_n = a$.

In particular, this theorem implies that if for all n

$$a \leqslant y_n \leqslant z_n$$

and it is known that $z_n \to a$, then also $y_n \to a$. Incidentally, this result can easily be proved directly.

The results *(1)*, *(2)* and *(3)* can easily be extended to the case of infinite limits.

39. Theorems on infinitesimals. In future arguments we may have to consider two (or more) variables simultaneously, connecting them by various arithmetical operations. Then, as before, we refer these operations to the corresponding values of the variables. For instance, speaking of a sum of two variables x_n and y_n ranging separately over the sequences of the values

$$x_1, x_2, x_3, \ldots, x_n, \ldots,$$

$$y_1, y_2, y_3, \ldots, y_n, \ldots,$$

we mean the variable $x_n + y_n$ taking the sequence of values

$$x_1 + y_1, x_2 + y_2, x_3 + y_3, \ldots, x_n + y_n, \ldots.$$

In proving theorems concerning the results of arithmetical operations over variables, the following two lemmas on infinitesimals will be useful.

LEMMA 1. *The sum of an arbitrary finite number of infinitesimals is also an infinitesimal quantity.*

We prove this result for the case of two infinitesimals α_n and β_n (the general result is proved in the same way).

Take an arbitrary number $\varepsilon > 0$. According to the definition of the infinitesimal, for any $\varepsilon/2$ we can find for the infinitesimal α_n a number N' such that for $n > N'$ we have

$$|\alpha_n| < \frac{\varepsilon}{2}.$$

Similarly, for the number β_n a number N'' can be found such that for $n > N''$ we have

$$|\beta_n| < \frac{\varepsilon}{2}.$$

If we take a positive integer N greater than N' and N'', then for $n > N$ both inequalities are satisfied simultaneously; hence

$$|\alpha_n + \beta_n| \leqslant |\alpha_n| + |\beta_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, the quantity $\alpha_n + \beta_n$ is infinitesimal.

LEMMA 2. *The product of a bounded variable x_n and an infinitesimal α_n is an infinitesimal quantity.*

For all values of n, let

$$m \leqslant x_n \leqslant M.$$

Denoting by L the greater of the absolute values $|m|, |M|$ we have

$$-L \leqslant m \leqslant x_n \leqslant M \leqslant L \quad \text{or} \quad |x_n| \leqslant L.$$

If an arbitrary number $\varepsilon > 0$ is given, then for the number ε/L, for the infinitesimal α_n, a number N can be found such that for $n > N$ we have

$$|\alpha_n| < \frac{\varepsilon}{L}.$$

For these values of n we obviously have

$$|x_n \cdot \alpha_n| = |x_n| \cdot |\alpha_n| < L \cdot \frac{\varepsilon}{L} = \varepsilon.$$

This implies that $x_n \cdot \alpha_n$ is an infinitesimal.

40. Arithmetical operations on variables. The following results are important in that by applying them in practical applications it will become unnecessary to return every time to the basic definition of the concept of a limit, connected with finding N for a given ε, and so on. This considerably simplifies the computation of limits.

(1) If the variables x_n and y_n have finite limits

$$\lim x_n = a, \quad \lim y_n = b,$$

then also their sum (difference) has a finite limit, and

$$\lim (x_n \pm y_n) = a \pm b.$$

The condition of the theorem implies that

$$x_n = a + \alpha_n, \quad y_n = b + \beta_n, \tag{3}$$

where α_n and β_n are infinitesimals. Then

$$x_n \pm y_n = (a \pm b) + (\alpha_n \pm \beta_n).$$

Here $\alpha_n \pm \beta_n$ is infinitesimal, by Lemma 1 of Sec. 39; consequently, the variable $x_n \pm y_n$ has the limit $a \pm b$, which completes the proof.

This result and its proof can be extended to the case of an arbitrary finite number of terms.

(2) If the variables x_n and y_n have finite limits

$$\lim x_n = a, \quad \lim y_n = b,$$

then their product also has a finite limit and

$$\lim x_n y_n = ab.$$

Using the same equations (3) we now have

$$x_n y_n = ab + (a\beta_n + b\alpha_n + \alpha_n \beta_n).$$

The expression in parenthesis, by Lemmas 1 and 2 of Sec. 39, is an infinitesimal. This implies that the variable $x_n y_n$ has in fact the limit ab.

This result can be extended to an arbitrary finite number of factors (for instance, by the method of mathematical induction).

(3) If the variables x_n and y_n have finite limits

$$\lim x_n = a, \quad \lim y_n = b,$$

and b is distinct from zero, then their ratio also has a finite limit, namely

$$\lim \frac{x_n}{y_n} = \frac{a}{b}.$$

For instance, let $b>0$; introduce between b and zero a number r. Then, by Sec. 36, *(1)* for sufficiently large n

$$y_n > r > 0,$$

so that in any case $y_n \neq 0$. Confining ourselves to those values of n for which this is true we know that the ratio x_n/y_n certainly has a meaning.

Using relations (3) again we have

$$\frac{x_n}{y_n} - \frac{a}{b} = \frac{a+\alpha_n}{b+\beta_n} - \frac{a}{b} = \frac{1}{by_n}(b\alpha_n - a\beta_n).$$

In view of Lemmas 1 and 2, the expression in parenthesis is an infinitesimal. From the initial statement we see that its factor is bounded

$$0 < \frac{1}{by_n} < \frac{1}{br}.$$

Consequently, by Lemma 2 the whole product on the right is infinitesimal, but it represents the difference between the variable x_n/y_n and the number a/b. Thus, the limit of x_n/y_n is a/b; this completes the proof.

41. Indefinite expressions. In the preceding subsection we considered the expressions

$$x_n \pm y_n, \quad x_n y_n, \quad \frac{x_n}{y_n} \tag{4}$$

and, assuming that the variables x_n and y_n tend to finite limits (in the case of a quotient the limit of y_n should be different from zero) we established limits for each of these expressions.

We omitted the cases when the limits of x_n and y_n (either one or both) were infinite or, in the case of a quotient, when the limit of the denominator was zero. We shall here consider only four of these cases—those which are of some importance and have an interesting feature.

(1) Consider first the quotient x_n/y_n and assume that both variables x_n and y_n tend simultaneously to zero. Here for the first time we encounter the exceptional circumstance: we know the limits of x_n and y_n, but we cannot make any general statement about the limit of their ratio without knowing the functions of n themselves.

This limit, depending on the particular law of variation of these variables, can have various values and may even not exist. The following simple examples clarify this statement.

Let for instance $x_n = 1/n^2$ and $y_n = 1/n$; both variables tend to zero. Their ratio $x_n/y_n = 1/n$ also tends to zero. If, conversely, we set $x_n = 1/n$, $y_n = 1/n^2$, then, although they tend to zero, now their ratio $x_n/y_n = n$ tends to $+\infty$. Taking now an arbitrary non-vanishing number a and constructing two infinitesimals $x_n = a/n$ and $y_n = 1/n$ we observe that their ratio has the limit a (since it identically equals a).

Finally, if $x_n = (-1)^{n+1}/n$, $y_n = 1/n$ (both limits are zero), then the ratio $x_n/y_n = (-1)^{n+1}$ has no limit at all.

Thus, the knowledge of the limits of the variables x_n and y_n, only in the considered cases is not sufficient for investigating their ratio: it is also necessary to know the functions themselves, i.e. their law of variation with n, and it is necessary to investigate directly the ratio x_n/y_n. In order to describe this peculiarity we say that when $x_n \to 0$ and $y_n \to 0$ the expression x_n/y_n represents an *indeterminate form of the type* 0/0.

(2) When $x_n \to \pm\infty$ and $y_n \to \pm\infty$ simultaneously, a similar case occurs. Without knowing the functions themselves no general statement can be made about the behaviour of their ratio as n tends to ∞. This fact can be illustrated by examples analogous to those quoted in *(1)*:

$$x_n = n \to \infty, \qquad y_n = n^2 \to \infty, \qquad \frac{x_n}{y_n} = \frac{1}{n} \to 0;$$

$$x_n = n^2 \to \infty; \qquad y_n = n \to \infty, \qquad \frac{x_n}{y_n} = n \to \infty;$$

$$x_n = an \to \pm\infty \ (a \gtrless 0), \qquad y_n = n \to \infty, \qquad \frac{x_n}{y_n} = a \to a.$$

Now, the expression

$$x_n = [2 + (-1)^{n+1}]\, n \to \infty, \qquad y_n = n \to \infty,$$

$$\frac{x_n}{y_n} = 2 + (-1)^{n+1}$$

has no limit at all.

In this case it is said that the expression x_n/y_n represents an *indeterminate form of the type* ∞/∞.

Consider now the product $x_n y_n$.

(3) If x_n tends to zero while y_n tends to $\pm\infty$, then considering the behaviour of the product $x_n y_n$ we encounter the same phenomenon as in *(1)* and *(2)*. The following examples illustrate this:

$$x_n = \frac{1}{n^2} \to 0, \qquad y_n = n \to \infty, \qquad x_n y_n = \frac{1}{n} \to 0;$$

$$x_n = \frac{1}{n} \to 0, \qquad y_n = n^2 \to \infty, \qquad x_n y_n = n \to \infty;$$

$$x_n = \frac{a}{n} \to 0 \ (a \gtrless 0), \qquad y_n = n \to \infty, \qquad x_n y_n = a \to a.$$

The expression

$$x_n = \frac{(-1)^{n+1}}{n} \to 0, \qquad y_n = n \to \infty, \qquad x_n y_n = (-1)^{n+1}$$

has no limit at all.

In this connection, when $x_n \to 0$ and $y_n \to \infty$ it is said that the expression $x_n y_n$ represents an *indeterminate form of the type* $0 \cdot \infty$.

Finally, consider the sum $x_n + y_n$.

(4) Here we obtain an exceptional case when x_n and y_n tend to infinities of different signs: again, we cannot say anything about the sum $x_n + y_n$ without knowing the functions x_n and y_n themselves. The various possibilities of this case are illustrated by the following examples:

$$x_n = 2n \to +\infty, \qquad y_n = -n \to -\infty, \qquad x_n + y_n = n \to +\infty;$$

$$x_n = n \to +\infty, \qquad y_n = -2n \to -\infty, \qquad x_n + y_n = -n \to -\infty;$$

$$x_n = n + a \to +\infty, \qquad y_n = -n \to -\infty, \qquad x_n + y_n = a \to a.$$

The expression

$$x_n = n + (-1)^{n+1} \to +\infty, \qquad y_n = -n \to -\infty,$$
$$x_n + y_n = (-1)^{n+1}$$

has no limit at all.

Owing to this, when $x_n \to +\infty$ and $y_n \to -\infty$ it is said that the expression $x_n + y_n$ represents an *indeterminate form of the type* $\infty - \infty$.

Thus, it is not always possible to determine the limits of arithmetical expressions (4) having the limits of the variables x_n and y_n. We have found four cases when this is certainly impossible: the indeterminancies of the forms

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0\cdot\infty, \quad \infty-\infty\dagger.$$

In these cases we have to investigate the required expressions directly from the laws of variation of x_n and y_n. This kind of investigation has been called the solution of indeterminancies. In numerous cases it is not as simple as in the above examples.

42. Extension to the case of a function of an arbitrary variable. We now make a remark concerning the general case. Since we have in mind theorems in which the variables are connected by the equality sign, the inequality sign or arithmetical operations, we first of all stipulate that by such signs connecting two or more functions $f(x)$, $g(x)$, ... (defined in one domain $\mathfrak{X}$) we always understand that their values correspond to the same value of x.

All these results could be proved in a way similar to that of Sec. 37 but it should be remembered that, in fact, this is unnecessary. If we define the limit of a function from the "standpoint of sequence", then, since for variables depending on the index n the results are already proved, they are also valid for the general case of function.

For instance, let us consider the results *(1)*, *(2)* and *(3)* of Sec. 40.

Suppose that we are given two functions $f(x)$ and $g(x)$ in the domain $\mathfrak{X}$ (with the point of condensation a) and assume that as x tends to a they have finite limits

$$\lim f(x) = A, \quad \lim g(x) = B.$$

Then the functions

$$f(x)\pm g(x), \quad f(x)\cdot g(x), \quad \frac{f(x)}{g(x)} \tag{5}$$

also have finite limits (in the case of a quotient assuming that $B \neq 0$), namely

$$A\pm B, \quad A\cdot B, \quad \frac{A}{B}.$$

† Of course, these symbols are devoid of any numerical meaning. Each of them is only a brief characteristic for the expression of the corresponding type of indeterminancy.

"In the language of sequences" these relations are read as follows: if $\{x_n\}$ is an arbitrary sequence (all the elements of which are distinct from a) of values of x from $\mathfrak{X}$ having the limit a, then

$$f(x_n) \to A, \quad g(x_n) \to B.$$

If to these two functions of positive integral argument n we apply the above results, we at once obtain

$$\lim[f(x_n) \pm g(x_n)] = A \pm B, \quad \lim f(x_n) \cdot g(x_n) = A \cdot B,$$

$$\lim \frac{f(x_n)}{g(x_n)} = \frac{A}{B},$$

and this ("in the language of sequences") expresses the fact that was to be proved†.

In the same way we extend to the general case the statements of Sec. 41 concerning the "indefinite expressions" characterized by the symbols

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0 \cdot \infty, \quad \infty - \infty.$$

As in the simplest case when we dealt with functions of positive integral argument, it is insufficient to only know the limits of the functions $f(x)$ and $g(x)$ when considering the above indeterminancies; we now have to take into account the law of variation itself. Examples of the solution of indeterminancies will be found in the next subsection.

We shall return to this problem in § 3 of Chapter 7 where general methods will be given for solving indeterminancies by the methods of differential calculus.

43. Examples. *(1)* Let $p(x)$ be a polynomial in x with constant coefficients

$$p(x) = a_0 x^k + a_1 x^{k-1} + \ldots + a_{k-1} x + a_k \quad (a_0 \neq 0).$$

We seek its limit as $x \to +\infty$. If all the coefficients of the terms of the polynomial were positive (negative) it would at once be clear that the limit of $p(x)$ is $+\infty$ $(-\infty)$. In the case of coefficients of various signs, however, some terms tend to $+\infty$ while others tend to $-\infty$ and we are faced with an indeterminancy of the form $+\infty - \infty$.

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† In the case of the quotient we may remark [as was done for y_n in Sec. 40, *(3)*], that for x sufficiently close to a the denominator $g(x) \neq 0$, so that the fraction $f(x)/g(x)$ has sense, at least for these values of x.

To solve it let us represent $p(x)$ in the form

$$p(x) = x^k\left(a_0 + \frac{a_1}{x} + \ldots + \frac{a_{k-1}}{x^{k-1}} + \frac{a_k}{x^k}\right).$$

Since all the terms in parenthesis, beginning from the second, are infinitesimal when x increases indefinitely, the expression in parenthesis has the limit $a_0 \neq 0$; the first factor tends to $+\infty$. Thus the whole expression tends to $+\infty$ or $-\infty$, depending on the sign of a_0.

In particular, the same result is obtained if instead of a continuously changing variable x we consider a positive integer n.

We leave it to the reader to find the limit of $p(x)$ when $x \to -\infty$ (taking now into account the parity or imparity of the exponent k). In all cases the limit of the polynomial $p(x)$ coincides with the limit of the term a_0x^k with the highest power.

This method of removing the "indeterminancies" by transforming the expression will frequently be employed.

(2) If $q(x)$ is a similar polynomial

$$q(x) = b_0x^l + b_1x^{l-1} + \ldots + b_{l-1}x + b_l \quad (b_0 \neq 0),$$

then the quotient $p(x)/q(x)$ has an indeterminancy of the type ∞/∞ for $x \to +\infty$.

Transforming each polynomial in a similar way to that of Example *(1)* we obtain

$$\frac{p(x)}{q(x)} = x^{k-l}\,\frac{a_0 + \frac{a_1}{x} + \ldots + \frac{a_k}{x^k}}{b_0 + \frac{b_1}{x} + \ldots + \frac{b_l}{x^l}}.$$

The second factor has a finite limit $a_0/b_0 \neq 0$. If the powers of both polynomials are equal, $k = l$, the limit of the ratio $p(x)/q(x)$ is also a_0/b_0. When $k > l$ the first factor tends to infinity for $x \to +\infty$ so that $p(x)/q(x)$ tends to $\pm\infty$ (depending on the sign of a_0/b_0). Finally, when $k < l$ the limit is zero. Here again we can substitute the positive integer n instead of x.

It is also easy to establish the limit of $p(x)/q(x)$ when $x \to -\infty$. In all cases the limit of a ratio of polynomials is equal to the limit of the ratio of the terms with the highest powers.

(3) Find the area Q of the figure OPM generated by the part OM of the parabola $y = ax^2$ $(a > 0)$, the segment OP of the x-axis and the segment OM (Fig. 24).

Divide the segment OP into n equal parts and construct on the latter a sequence of rectangles with defect and with excess. The areas Q_n and Q_n' of these step figures differ by the area $y(x/n)$ of the greatest rectangle. Hence, the difference $Q_n' - Q_n \to 0$ (as $n \to \infty$) and since

$$Q_n < Q < Q_n',$$

we obviously have

$$Q = \lim Q_n = \lim Q_n'.$$

Since the heights of the rectangles are the ordinates of the points of the parabola with the abscissae

$$\frac{1}{n}x,\ \frac{2}{n}x,\ \ldots,\ \frac{n}{n}x = x,$$

and according to the equation of the curve their magnitude is

$$a\frac{1}{n^2}x^2,\ a\frac{2^2}{n^2}x^2,\ \ldots,\ a\frac{n^2}{n^2}x^2,$$

respectively, we obtain for Q_n' the expression†

$$Q_n' = \frac{ax^2}{n^2}(1^2+2^2+\ldots+n^2)\frac{x}{n} = \frac{ax^3}{6}\frac{(n+1)(2n+1)}{n^2}.$$

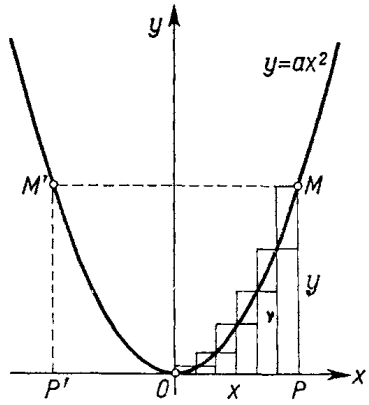

FIG. 24.

Hence, making use of Example *(2)*,

$$Q = \lim Q_n' = \frac{ax^3}{3} = \frac{xy}{3}.$$

From this it is easy to find that the area of the parabolic segment $M'OM$ is equal to $(4/3)xy$, i.e. to two-thirds of the area of the described rectangle (this result was known to Archimedes‡).

Remark. The general definition of the area of a curvilinear figure will be given in Chapter 12; there the method applied now for the computation of the area will be generalized to other curvilinear figures [Sec. 196].

(4) Find the limits of the variables

$$x_n = \frac{n}{\sqrt{(n^2+n)}},\qquad z_n = \frac{n}{\sqrt{(n^2+1)}}$$

† We make use here of the well-known formula for the sum of the squares of the first n positive integers.

‡ Archimedes—the greatest ancient mathematician (*c.* 287–212 B.C.).

and finally

$$y_n = \frac{1}{\sqrt{(n^2+1)}} + \frac{1}{\sqrt{(n^2+2)}} + \ldots + \frac{1}{\sqrt{(n^2+n)}}.$$

The expressions x_n and z_n have indeterminancies of the type ∞/∞ (since both roots are greater than n, they tend to infinity). Let us make a transformation, dividing the numerator and denominator by n:

$$x_n = \frac{1}{\sqrt{\left(1+\frac{1}{n}\right)}}, \quad z_n = \frac{1}{\sqrt{\left(1+\frac{1}{n^2}\right)}}.$$

Since both roots in the denominator have the limit unity†, then $x_n \to 1$ and $z_n \to 1$.

The expression for y_n is of a special form: every term of this sum depends on n but their number also increases with n. Since every term is smaller than the first and greater than the last we have

$$\frac{n}{\sqrt{(n^2+n)}} < y_n < \frac{n}{\sqrt{(n^2+1)}}, \quad \text{i.e.} \quad x_n < y_n < z_n.$$

But (in accordance with the proved result) the variables x_n and z_n tend to a common limit, which is unity; consequently, by result *(3)* of Sec. 38 y_n tends to the same limit.

(5) Let us return to the function $f(x)$ considered in Sec. 18, *(3)* and defined by three distinct formulae for various x. Now set:

$$f(x) = \lim_{n\to\infty} \frac{x^{2n}-1}{x^{2n}+1} \quad \text{for all } x.$$

If $|x| > 1$ we have here an indeterminancy of the type ∞/∞ which can easily be solved by dividing the numerator and denominator by x^{2n}; we obtain $f(x) = 1$. When $|x| < 1$ it is evident that $x^{2n} \to 0$ and $f(x) = -1$. Finally when $x = \pm 1$ the numerator of the fraction is always equal to zero and so $f(x) = 0$. This is exactly the same function but now given by one formula.

(6) $\lim\limits_{x\to 0} \dfrac{\sqrt{(1+x)}-1}{x} = \dfrac{1}{2}.$

In fact,

$$\frac{\sqrt{(1+x)}-1}{x} = \frac{1}{\sqrt{(1+x)}+1};$$

but

$$1-|x| < \sqrt{(1+x)} < 1+|x|,$$

† For instance, for the first root this follows from the inequalities

$$1 < \sqrt{\left(1+\frac{1}{n}\right)} < 1+\frac{1}{n} \quad \text{[Sec. 38, } (3)\text{]}.$$

so that

$$\lim_{x\to 0} \sqrt[n]{(1+x)} = 1,$$

which implies the required result.

(7) The limit [Sec. 34, *(5)*]

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

is frequently used for finding other limits.

$$\text{(a)} \quad \lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2} \qquad \left(\frac{0}{0}\right).$$

Evidently

$$\frac{1-\cos x}{x^2} = \frac{2\sin^2 \frac{x}{2}}{x^2} = \frac{1}{2}\left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2;$$

since the expression in parenthesis tends to unity, the general limit is 1/2.

$$\text{(b)} \quad \lim_{x\to 0} \frac{\tan x - \sin x}{x^3} = \frac{1}{2} \qquad \left(\frac{0}{0}\right).$$

Here again a transformation leads to the previously examined limits

$$\frac{\tan x - \sin x}{x^3} = \frac{1}{\cos x} \cdot \frac{\sin x}{x} \cdot \frac{1-\cos x}{x^2}.$$

Observe that $\cos x \to 1$ when $x \to 0$, which follows, for instance, from the preceding result.

§ 3. Monotonic functions

44. Limit of a monotonic function of a positive integral argument. Theorems on the existence of limits of functions considered so far have the following property: assuming that the limits of given functions exist we established the existence of limits for other functions in some way connected with the given ones. The problem of finding criteria for the existence of a finite limit without reference to other functions has so far not been considered. We shall solve this problem in general in § 4; here, however, we consider one simple and important class of functions for which the problem can easily be solved, and as before we begin from the simplest example, i.e. from a function x_n of positive integral argument.

The variable x_n is called *increasing* if

$$x_1 < x_2 < \ldots < x_n < x_{n+1} < \ldots,$$

i.e. if from $n' > n$ it follows that $x_{n'} > x_n$.

It is called *non-decreasing* if

$$x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n \leqslant x_{n+1} \leqslant \ldots,$$

i.e. if $n' > n$ implies only $x_{n'} \geqslant x_n$. In the latter case the variable can also be called *increasing* if this term be understood in a wider sense.

Similarly we establish the concept of decreasing (in the narrow or wide sense of the word) function of n: this is the variable for which

$$x_1 > x_2 > \ldots > x_n > x_{n+1} > \ldots$$

or

$$x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n \geqslant x_{n+1} \geqslant \ldots,$$

respectively. Thus, it follows from $n' > n$ (depending on the case considered) that $x_{n'} < x_n$ or only $x_{n'} \leqslant x_n$.

We call variables of any one of these types *monotonic*. It is usually said about such a variable that it is "monotonically increasing" or "monotonically decreasing".

We also use the expression increasing or decreasing (in the respective cases) to describe the sequence

$$x_1, x_2, x_3, \ldots, x_n, \ldots$$

where the variable x_n is increasing or decreasing.

For monotonic variables we have the following:

THEOREM. *Consider a monotonically increasing variable x_n. If it is bounded above*

$$x_n \leqslant M \quad (M = \text{const}, n = 1, 2, 3, \ldots),$$

then it necessarily has a finite limit; otherwise it tends to $+\infty$.

Similarly, a monotonically decreasing variable x_n also always has a limit. It is finite if x_n is bounded below; otherwise the limit is $-\infty$†.

† It is easily observed that all inferences remain valid also for a variable x_n which is monotonic only for sufficiently large n (since, without affecting the limit of the variable, an arbitrary number of its first values can be omitted). In the statement of the theorem, instead of monotonic x_n we could speak of a monotonic sequence.

Proof. We confine ourselves to the case of *increasing* (in the wider sense of the word) variable x_n (the case of a decreasing variable can be treated in the same way).

Assume first that this variable is bounded from above. Then, by the theorem of Sec. 6, for the set $\{x_n\}$ of its values there should exist (and be finite) the least upper bound

$$a = \sup \{x_n\};$$

we shall show that this number is the limit of the variable x_n.

In fact, let us recall the characteristic properties of the upper bound [Sec. 6]. First, for all values of n we have

$$x_n \leqslant a.$$

Secondly, for any number $\varepsilon > 0$ a value, say x_N, can be found for our variable, such that it exceeds the number $a - \varepsilon$,

$$x_N > a - \varepsilon.$$

Since in view of the monotonicity of the variable x_n (this is the first time we employ this property), for $n > N$ we have $x_n \geqslant x_N$, i.e. certainly $x_n > a - \varepsilon$; for these values of the number n we obtain the inequalities

$$0 \leqslant a - x_n < \varepsilon, \quad \text{hence} \quad |x_n - a| < \varepsilon,$$

which imply that $\lim x_n = a$.

Suppose now that the variable x_n is not bounded above. Then for arbitrarily large $\mathrm{E} > 0$, at least one value of the variable can be found which is greater than E; denote it by x_N, i.e. $x_N > \mathrm{E}$. In view of the monotonicity of the variable x_n for $n > N$ we certainly have

$$x_n > \mathrm{E},$$

and this implies that $\lim x_n = +\infty$.

Remark. The existence of a finite limit for a bounded monotonic variable was regarded in the first half of the nineteenth century as an obvious fact. The necessity of a precise proof of this statement, which is of fundamental importance, was in fact one of the reasons for creating an arithmetical theory of irrational numbers. Observe that the above statement is equivalent to the property of continuity of the set of real numbers [Sec. 5].

We now consider some examples of the above theorem.

45. Examples. *(1)* Consider the expression (assuming $c>0$)

$$x_n = \frac{c^n}{n!},$$

where $n! = 1\cdot 2\cdot \ldots \cdot n$ (for $c>1$ it is an indeterminant form of the type ∞/∞).

Since

$$x_{n+1} = \frac{c}{n+1}x_n,$$

then, provided that $n>c-1$, the variable is decreasing; at the same time it is bounded below, for instance by zero. Consequently, according to the theorem, the variable x_n has a *finite* limit which we shall denote by a.

To find it, we pass to the limit in the above relation; since x_{n+1} ranges over the same sequence of values as x_n (other than the first term) and has the same limit a we obtain

$$a = 0\cdot a,$$

whence $a = 0$ and finally

$$\lim \frac{c^n}{n!} = 0.$$

(2) Assuming again that $c>0$ we now define x_n as follows:

$$x_1 = \sqrt{c}, \quad x_2 = \sqrt{(c+\sqrt{c})}, \quad x_3 = \sqrt{[c+\sqrt{(c+\sqrt{c})}]}, \ldots$$

and in general

$$x_n = \underbrace{\sqrt{[c+(\sqrt{c}+\ldots+\sqrt{c})]}}_{n \text{ times}}.$$

Thus, x_{n+1} is obtained from x_n by the formula

$$x_{n+1} = \sqrt{(c+x_n)}.$$

Clearly, the variable x_n increases monotonically. At the same time it is bounded above, for instance by the number $\sqrt{c}+1$. In fact, $x_1 = \sqrt{c}$ is smaller than this number; if we now assume that some value of $x_n < \sqrt{c}+1$, then for the next value we obtain

$$x_{n+1} < \sqrt{(c+\sqrt{c}+1)} < \sqrt{(c+2\sqrt{c}+1)} = \sqrt{c}+1.$$

Thus, our statement is proved by mathematical induction.

According to the fundamental theorem the variable x_n has a finite limit a. To find it let us pass to the limit in the relation

$$x_{n+1}^2 = c + x_n;$$

thus we find that a satisfies the quadratic equation

$$a^2 = c + a.$$

This equation has roots of different signs; but the required limit a cannot be negative and consequently it is equal to the positive root

$$a = \frac{\sqrt{(4c+1)}+1}{2}\dagger.$$

Both examples lead to the following remark. The theorem proved in Sec. 44 is a typical "existence theorem": it establishes the fact of existence of a limit but no method for its computation is given. Nevertheless it is of great importance. In the first place, in theoretical problems frequently only the existence of the limit is relevant; secondly, preliminary proof of the existence of a limit is important since it paves the way to actually calculating it. Thus, in the above examples the knowledge of the existence of the limit made it possible, by passing to the limit in certain relations, to establish the value of the required limit.

46. A lemma on imbedded intervals. We shall now consider relations between two monotonic variables varying in "opposite" directions.

Consider a monotonically increasing variable x_n and a monotonically decreasing variable y_n, such that

$$x_n < y_n \tag{1}$$

for all n. If their difference $y_n - x_n$ tends to zero, both variables have the same finite limit

$$c = \lim x_n = \lim y_n.$$

In fact, for all values of n we have $y_n \leqslant y_1$ and hence, by (1), also $x_n < y_1$ $(n = 1, 2, 3, \ldots)$. The increasing variable x_n turns out to be bounded above and therefore has a finite limit

$$c = \lim x_n.$$

Similarly, for the decreasing variable y_n we have

$$y_n > x_n \geqslant x_1,$$

and hence it also tends to the finite limit

$$c' = \lim y_n.$$

---

† This interesting example belongs actually to Jacob Bernoulli, who considered it in the form of computation of the expression

$$\sqrt{\{c+\sqrt{[c+\sqrt{(c+\ldots}},$$ etc., to infinity.

Now, by result *(1)* of Sec. 40 the difference between the two limits

$$c' - c = \lim (y_n - x_n),$$

i.e. by assumption it is zero, whence $c' = c$, which completes the proof.

This statement can be put in another form, more frequently used.

We say that the interval $[a', b']$ is contained in the interval $[a, b]$, or *is imbedded* in it, if all points of the former interval belong to the latter, or, equivalently, if

$$a \leqslant a' < b' \leqslant b.$$

The geometrical meaning of the above statement is clear.

Consider an infinite sequence of intervals imbedded in one another

$$[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n], \ldots,$$

so that each interval is contained in the preceding one, and the lengths of these intervals tend to zero with increasing n:

$$\lim (b_n - a_n) = 0.$$

Then the ends a_n and b_n of these intervals (from different sides) tend to the common limit

$$c = \lim a_n = \lim b_n.$$

This is only another statement of the above theorem; by assumption

$$a_n \leqslant a_{n+1} < b_{n+1} \leqslant b_n,$$

so that the left-hand end a_n and the right-hand end b_n of the nth interval play the roles of the monotonic variables x_n and y_n.

In future we shall frequently require this result, which is called "the lemma on imbedded intervals".

47. The limit of a monotonic function in the general case. We now proceed to consider the function $f(x)$ of an arbitrary variable. Here again the problem of the existence of the limit of the function

$$\lim_{x \to a} f(x)$$

is solved very simply for functions of a particular type—those constituting a generalization of the concept of a monotonic variable x_n [Sec. 44].

Suppose that the function $f(x)$ is defined in a domain $\mathfrak{X} = \{x\}$. The function is said to be increasing (decreasing) in this domain if for any pair of its values x and x' it follows from $x' > x$ that

$$f(x') > f(x) \; [f(x') < f(x)].$$

If now it follows from $x' > x$ that

$$f(x') \geqslant f(x) \; [f(x') \leqslant f(x)],$$

the function is called *non-decreasing* (*non-increasing*). Sometimes it is more convenient, as before, to call the function *increasing* (*decreasing*) in the wider sense of the word.

Functions of these types have the general name monotonic functions. For a monotonic function there is a theorem which is analogous to that on the monotonic variable x_n depending on n, established in Sec. 44.

THEOREM. *Suppose that the function $f(x)$ increases monotonically, in either sense, in a domain $\mathfrak{X}$ having a point of condensation a greater than all values of x (it can be finite or equal to $+\infty$). If the function is bounded above,*

$$f(x) \leqslant M \quad (\textit{for all } x \textit{ in } \mathfrak{X}),$$

then, for $x \to a$, the function has a finite limit; otherwise it tends to $+\infty$.

Proof. First assume that the function $f(x)$ is bounded above, i.e. the set $\{f(x)\}$ of values of the function is bounded above, these values corresponding to the variation of x in $\mathfrak{X}$. Then for this set there exists [Sec. 6] a finite least upper bound A. We now prove that this number A is the required limit.

First of all, for all values of x

$$f(x) \leqslant A.$$

Further, taking an arbitrary number $\varepsilon > 0$, by a property of the least upper bound we can find a value of $x' < a$ such that $f(x') > A - \varepsilon$. In view of the monotonicity of the function, for $x > x'$ we certainly have $f(x) > A - \varepsilon$ and hence for these values of x the inequality

$$|f(x) - A| < \varepsilon$$

holds.

This proves our statement; it is only necessary for a finite a to take $\delta = a - x'$ (so that the inequality $x > x'$ can be written in the form $x > a - \delta$); for $a = +\infty$ we take $\Delta = x'$.

If the function $f(x)$ is not bounded above, then for any number E a value x' can be found such that $f(x') > E$; then for $x > x'$ we certainly have $f(x) > E$, and so on.

We leave it to the reader to transform this theorem to the case when the limit a is smaller than all values of x, i.e. the case of a monotonically decreasing function.

Clearly, the theorem on the monotonic variable x_n in Sec. 44 is simply a particular case of this theorem. The independent variable in the former case was the number n and the domain of variability was the sequence of positive integers $\mathcal{N} = \{n\}$ with the point of condensation $+\infty$.

In what follows we shall usually take as the domain $\mathcal{X}$, over which the function $f(x)$ is examined, the *continuous interval* $[a', a)$ where $a' < a$ and a is a finite number or $+\infty$, or else the interval $(a, a']$ where $a' > a$ and a is a finite number or $-\infty$.

§ 4. The number e

48. The number e defined as the limit of a sequence. We shall employ here the method of passage to the limit to define a new number, which has so far not been encountered and which is of considerable importance both in analysis itself and for its applications.

Consider the variable

$$x_n = \left(1 + \frac{1}{n}\right)^n$$

to which we shall try to apply the theorem of Sec. 44.

Since the expression $(1 + 1/n)$ decreases when the exponent n increases, the "monotonic" nature of this variable is not obvious. To show, however, that it is monotonic let us apply the binomial expansion; thus

$$x_n = \left(1 + \frac{1}{n}\right)^n = 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{1 \cdot 2} \cdot \frac{1}{n^2}$$

$$+ \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} \frac{1}{n^3} + \ldots + \frac{n(n-1)\ldots(n-k+1)}{1 \cdot 2 \cdot \ldots \cdot k} \cdot \frac{1}{n^k} + \ldots$$

$$+ \frac{n(n-1)\ldots(n-n+1)}{1 \cdot 2 \cdot \ldots \cdot n} \cdot \frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \dots$$
$$+ \frac{1}{k!}\left(1 - \frac{1}{n}\right)\dots\left(1 - \frac{k-1}{n}\right) + \dots$$
$$+ \frac{1}{n!}\left(1 - \frac{1}{n}\right)\dots\left(1 - \frac{n-1}{n}\right). \qquad (1)$$

If we now pass from x_n to x_{n+1}, i.e. we increase n by unity, then first of all a new $(n+2)$th (*positive*) term appears, while each of the existing $(n+1)$ terms *increases*, since every factor in parenthesis of the form $1 - s/n$ is replaced by a greater factor $1 - s/(n+1)$. It follows that

$$x_{n+1} > x_n,$$

i.e. the variable x_n is *increasing*.

We now prove that this variable is also bounded above. By omitting from expression (1) all factors in the parenthesis we increase it; thus

$$x_n < 2 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} = y_n.$$

Further, replacing every factor in the denominator of the fractions (beginning from the third) by the number 2 we further increase the expression. Hence

$$y_n < 2 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}}.$$

But the progression (the first term of which is 1/2) has a sum which is smaller than unity, whence $y_n < 3$ and so $x_n < 3$.

This implies, according to the theorem of Sec. 44, that the variable x_n has a finite limit. Following Euler, it is denoted by the letter e. Thus we write

$$e = \lim\left(1 + \frac{1}{n}\right)^n.$$

The first 15 digits in its decimal expansion are

$$e = 2.71828\ 18284\ 59045\dots.$$

Although the sequence

$$x_1 = \left(1+\frac{1}{1}\right)^1 = 2; \quad x_2 = \left(1+\frac{1}{2}\right)^2 = 2.25;$$

$$x_3 = \left(1+\frac{1}{3}\right)^3 = 2.3703\ldots;$$

$$x_{100} = \left(1+\frac{1}{100}\right)^{100} = 2.7048\ldots;\ \ldots$$

does tend to the number e, the convergence is slow and it is inconvenient to use it for an approximate calculation of the number e. In the next subsection we give a better method for finding it, and we shall, incidentally, prove that e is an irrational number.

49. Approximate computation of the number e. We return to relation (1). If we fix k and, assuming $n>k$, we disregard all terms of the remainder after the $(k+1)$th term, we arrive at the relation

$$x_n > 2+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\ldots$$
$$+\frac{1}{k!}\left(1-\frac{1}{n}\right)\ldots\left(1-\frac{k-1}{n}\right).$$

Increasing n to infinity we pass to the limit; since all parentheses have the limit unity, we obtain

$$e \geqslant 2+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{k!} = y_k.$$

This inequality holds for any positive integer k. Thus we have

$$x_n < y_n \leqslant e,$$

which clearly implies (by the result *(3)* of Sec. 38) that also

$$\lim y_n = e.$$

The variable y_n is much more convenient for an approximate computation of the number e than x_n. Let us estimate the nearness of y_n to e. For this purpose first consider the difference between any value y_{n+m} $(m = 1, 2, 3, \ldots)$, following y_n, and y_n itself. We have

$$y_{n+m}-y_n = \frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\ldots+\frac{1}{(n+m)!}$$
$$= \frac{1}{(n+1)!}\left\{1+\frac{1}{n+2}+\frac{1}{(n+2)(n+3)}+\ldots+\frac{1}{(n+2)(n+3)\ldots(n+m)}\right\}.$$

If in the brackets $\{...\}$ we replace all factors in the denominators of the fractions by $n+2$ we obtain the inequality

$$y_{n+m}-y_n<\frac{1}{(n+1)!}\left\{1+\frac{1}{n+2}+\frac{1}{(n+2)^2}+\ldots+\frac{1}{(n+2)^{m-1}}\right\},$$

so that replacing the brackets by the sum of the infinite progression we find that

$$y_{n+m}-y_n<\frac{1}{(n+1)!}\cdot\frac{n+2}{n+1}.$$

Fixing n we let m tend to infinity; the variable y_{n+m} (the number of which is m) takes the sequence of values

$$y_{n+1}, y_{n+2}, \ldots, y_{n+m}, \ldots,$$

which evidently converges to e. Hence in the limit we obtain

$$e-y_n\leqslant\frac{1}{(n+1)!}\cdot\frac{n+2}{n+1}$$

or finally

$$0<e-y_n<\frac{1}{n!\,n}\dagger.$$

If we denote the ratio of the difference $e-y_n$ to the number $1/(n!\,n)$ by θ (evidently, it lies between zero and unity) we can also write

$$e-y_n=\frac{\theta}{n!\,n}.$$

Replacing here y_n by the explicit expression we arrive at the important formula

$$e=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}+\frac{\theta}{n!\,n}, \tag{2}$$

from which we can compute the value of e. Omitting the last "additional" term and replacing each of the remaining terms by its decimal approximation we obtain the required value of e.

We first compute e, by means of formula (2), to the accuracy $1/10^4$. First we have to establish the number n (which is at our disposal) in order to obtain this accuracy.

---

† Because (as we can easily check) $\dfrac{n+2}{(n+1)^2}<\dfrac{1}{n}$.

Computing consecutively the values of $1/n!$, $n = 1, 2, 3, \ldots$ (see the accompanying table) we observe that for $n = 7$ the "additional" term of formula (2) has the value

$$\frac{\theta}{n!n} = \frac{\theta}{7!7} < 0.00003,$$

so that the error involved in omitting it is smaller than $1/10^4$. We therefore take $n = 7$. Now all remaining terms are replaced by decimal approximations stopping (to increase the accuracy) at the fifth decimal place, so that the absolute value of the error is smaller than half of unity on the fifth place, i.e. smaller than $1/(2 \cdot 10^5)$. The results of the computations are given in the table. Every approximate value is supplied with a sign (+ or −) indicating the sign of the correction which it would be necessary to add to re-establish the exact number.

	2.00000
$\frac{1}{2!} =$	0.50000
$\frac{1}{3!} =$	0.16667 −
$\frac{1}{4!} =$	0.04167 −
$\frac{1}{5!} =$	0.00833 +
$\frac{1}{6!} =$	0.00139 −
$\frac{1}{7!} =$	0.00020 −
	2.71826

Thus, we see that the error due to omitting the additional term is smaller than $3/10^5$. Now taking into account the errors (with their signs) due to stopping at the fifth place, it is readily seen that the total error in the derived approximate value of the number e lies between

$$-\frac{2}{10^5} \quad \text{and} \quad +\frac{3.5}{10^5}.$$

Hence the number e is contained between the fractions

$$2.71824 \quad \text{and} \quad 2.718295,$$

so that we can set

$$e = 2.7182 + 0.0001.$$

Observe that formula (2) can also be used to prove the *irrationality* of the number e.

Assuming the converse, suppose that e is equal to a rational fraction m/n; then if we write down formula (2) for this n we obtain

$$\frac{m}{n} = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} + \frac{\theta}{n!n} \quad (0 < \theta < 1).$$

Multiplying throughout this equation by $n!$, reducing the denominator of all fractions except the last one, we obtain on the left an integer and on the right an integer plus the fraction θ/n, which is impossible. The contradiction proves the assertion.

50. The basic formula for the number e. Natural logarithms. The number e in Sec. 48 was originally defined as the limit of a variable depending on a positive integral argument,

$$e = \lim\left(1 + \frac{1}{n}\right)^n. \tag{3}$$

We now proceed to establish a more general result

$$e = \lim_{x \to 0} (1+x)^{\frac{1}{x}}. \tag{4}$$

For this purpose [Sec. 35] it is sufficient to prove that the following relations are separately valid:

$$\lim_{x \to +0} (1+x)^{\frac{1}{x}} = e \quad \text{and} \quad \lim_{x \to -0} (1+x)^{\frac{1}{x}} = e. \tag{4a}$$

We now make use of the definition of limit "in the language of sequences" [Sec. 32].

Incidentally, if the limit (3), also regarded as the limit of a function of n, be considered "in the language of sequences", we arrive at the relation

$$\lim \left(1+\frac{1}{n_k}\right)^{n_k} = e, \tag{5}$$

regardless of the sequence $\{n_k\}$ of the positive integers increasing with the number k to infinity.

Now let x range over a sequence $\{x_k\}$ of *positive* values tending to zero; we may assume that all $x_k < 1$. Set $n_k = E(1/x_k)$ so that

$$n_k \leqslant \frac{1}{x_k} < n_k + 1 \quad \text{and} \quad n_k \to +\infty.$$

Since also

$$\frac{1}{n_k+1} < x_k \leqslant \frac{1}{n_k},$$

we have

$$\left(1+\frac{1}{n_k+1}\right)^{n_k} < (1+x_k)^{\frac{1}{x_k}} < \left(1+\frac{1}{n_k}\right)^{n_k+1}.$$

The two outside expressions can be transformed as follows:

$$\left(1+\frac{1}{n_k+1}\right)^{n_k} = \frac{\left(1+\frac{1}{n_k+1}\right)^{n_k+1}}{1+\frac{1}{n_k+1}},$$

$$\left(1+\frac{1}{n_k}\right)^{n_k+1} = \left(1+\frac{1}{n_k}\right)^{n_k}\left(1+\frac{1}{n_k}\right),$$

and, by (5)

$$\left(1+\frac{1}{n_k}\right)^{n_k}\to e, \quad \text{but also} \quad \left(1+\frac{1}{n_k+1}\right)^{n_k+1}\to e,$$

while it is evident that

$$1+\frac{1}{n_k}\to 1, \quad 1+\frac{1}{n_k+1}\to 1.$$

Thus, both the above expressions tend to the common limit e and hence [by the result *(3)* of Sec. 38] the expression between them also tends to e,

$$\lim(1+x_k)^{\frac{1}{x_k}}=e.$$

This completes the proof of the first relation (4a) "in the language of sequences".

To prove the second relation assume now that the sequence $\{x_k\}$ consists of negative values tending to zero; we suppose also that $x_k>-1$. If we now set $x_k=-y_k$, then

$$1>y_k>0, \quad y_k\to 0.$$

Obviously,

$$(1+x_k)^{\frac{1}{x_k}}=(1-y_k)^{-\frac{1}{y_k}}=\left(\frac{1}{1-y_k}\right)^{\frac{1}{y_k}}$$

$$=\left(1+\frac{y_k}{1-y_k}\right)^{\frac{1-y_k}{y_k}}\left(1+\frac{y_k}{1-y_k}\right).$$

Since, by what has just been proved, the first factor of the last expression tends to e, the limit of the second is unity, and the whole expression tends to e. Thus, formula (4) has been fully justified.

This remarkable property of the number e constitutes the base of all its applications. It is just this property that makes e so convenient for use as the base of a logarithm system. The logarithms with the base e are called natural and are denoted by the symbol log†; in theoretical investigations only natural logarithms are employed‡.

† Without a subscript, i.e. we write log instead of the fuller form $\log_e$. Sometimes the notation ln (from *logarithmus naturalis*) is used instead.

‡ These logarithms are sometimes erroneously called Napier's logarithms, in accordance with the name of a Scottish mathematician, Napier (1550–1617)—

Observe that the ordinary decimal logarithms are connected with the natural ones by the formula

$$\log_{10} x = \log x \cdot M,$$

where M is the transformation modulus equal to

$$M = \log_{10} e = \frac{1}{\log 10} = 0.434294 \ldots;$$

this result can easily be derived by taking logarithms with the base 10 in the identity

$$x = e^{\log x}.$$

§ 5. The principle of convergence

51. Partial sequences. Consider a sequence

$$x_1, x_2, x_3, \ldots, x_n, \ldots, x_{n'}, \ldots. \tag{1}$$

Consider also a *partial* sequence extracted from it:

$$x_{n_1}, x_{n_2}, x_{n_3}, \ldots, x_{n_k}, \ldots \tag{2}$$

where $\{n_k\}$ is a sequence of *increasing* positive integers

$$n_1 < n_2 < n_3 < \ldots < n_k < n_{k+1} < \ldots. \tag{3}$$

The independent variable taking consecutively all positive integral values is now not n but k; n_k is a function of k taking only positive integral values and evidently tending to infinity with increasing k.

If the sequence (1) has a definite limit a (finite or otherwise), then the partial sequence (2) has the same limit. If there is no definite limit for sequence (1) this does not rule out the possibility of the existence of a limit for some partial sequence.

Suppose, for instance, that $x_n = (-1)^{n+1}$; this variable has no limit. If however we assume that n ranges over only odd or only even values, then the partial sequences

$$x_1 = 1, \; x_3 = 1, \; \ldots, \; x_{2k-1} = 1, \ldots$$

---

the discoverer of logarithms. Napier himself had no idea about a base of a logarithm system, since he constructed them in a special way on an entirely different principle, but his logarithms correspond to logarithms the base of which is close to $1/e$. Logarithms of one of his contemporaries, a Swiss mathematician, Bourgi (1552–1632), have a base close to e.

and

$$x_2 = -1,\ x_4 = -1,\ \ldots,\ x_{2k} = -1,\ \ldots$$

have the limits 1 and -1, respectively.

In the case of an *unbounded* sequence (1) it is sometimes impossible to separate out any partial sequence (2) which has a finite limit (this is for instance the case when the sequence (1) tends to $\pm\infty$). However, for a finite sequence the following result, due to Bolzano and Weierstrass, is true:†

The BOLZANO–WEIERSTRASS LEMMA. *From any finite sequence* (1) *a partial sequence* (2), *which tends to a finite limit, can always be extracted.*

(This formulation does not prevent us from having *equal* numbers in the considered sequence; this possibility is often useful in applications.)

Proof. Let all numbers x_n be contained between the bounds a and b. Divide this interval $[a, b]$ into halves, then at least in one half there is an infinite set of elements of the considered sequence, since otherwise the whole interval $[a, b]$ would contain only a finite number of elements, which is impossible. Thus, let $[a_1, b_1]$ be the half contaning an infinite set of values of x_n (or either interval, if both halves contain an infinite set of numbers).

Similarly, from the interval $[a_1, b_1]$ we separate out the half $[a_2, b_2]$ that contains an infinite set of numbers x_n, etc. Continuing this process, in the kth step we obtain an interval $[a_k, b_k]$ which also contains an infinite set of the numbers x_n; and so on to infinity.

Each interval so constructed (starting from the second) is contained in the preceding one; moreover, the length of the kth interval is equal to

$$b_k - a_k = \frac{b-a}{2^k},$$

and tends to zero as k increases. Now applying the lemma on imbedded intervals (Sec. 46) we find that a_k and b_k tend to a common limit c.

We now construct, by means of induction, the partial sequence $\{x_{n_k}\}$ in the following way. For x_{n_1} take any (for instance, the first) element x_n of our sequence, which is contained in $[a_1, b_1]$. For x_{n_2} take any (for instance, the first) element x_n following x_{n_1} and contained

† Karl Weierstrass (1815–1879)—an outstanding German mathematician.

in $[a_2, b_2]$, and so on. In general, for x_{n_k} we take any (for instance, the first) element x_n following $x_{n_1}, x_{n_2}, \ldots, x_{n_{k-1}}$ and contained in $[a_k, b_k]$. Such a selection can be effected since every interval $[a_k, b_k]$ contains an infinite set of the numbers x_n, i.e. it contains elements x_n with arbitrarily large numbers n.

Furthermore, since

$$a_k \leqslant x_{n_k} \leqslant b_k \quad \text{and} \quad \lim a_k = \lim b_k = c,$$

by the result *(3)* of Sec. 38 we have also $\lim x_{n_k} = c$, which was to be proved.

The above method of consecutive division into halves of the considered intervals will be useful in other cases.

The Bolzano–Weierstrass lemma considerably simplifies the proofs of many difficult theorems, absorbing in a way the main difficulty of the reasoning. We shall employ it in the following subsection.

52. The condition of existence of a finite limit for a function of positive integral argument. Consider the variable x_n ranging over the values (1); let us now consider the problem of finding a general criterion for the existence of a finite limit for this variable (i.e. for the sequence). The definition of the limit cannot be used, since it contains the limit, the existence of which we want to prove. We require a criterion which would use only what we already are given, namely the sequence (1) of the values of the variable.

The above problem is solved by the following celebrated theorem which is due to Bolzano (1817) and Cauchy (1821); it is sometimes called the *principle of convergence.*

THEOREM. *In order that the variable x_n has a finite limit it is necessary and sufficient that for any number $\varepsilon > 0$ there exists a number N such that the inequality*

$$|x_n - x_{n'}| < \varepsilon \tag{4}$$

is valid, provided $n > N$ and $n' > N$.

As can be seen, the essence of the problem is that the values of the variable approach each other as their numbers increase. Let us now carry out the proof of its necessity.

Necessity. Let the variable x_n have a definite finite limit, say a. According to the very definition of the limit [Sec. 28], for any

$\varepsilon > 0$, a number N can be found such that for $n > N$ we have the inequality

$$|x_n - a| < \frac{\varepsilon}{2}.$$

Now take two numbers $n > N$ and $n' > N$; then we have simultaneously

$$|x_n - a| < \frac{\varepsilon}{2} \quad \text{and} \quad |a - x_{n'}| < \frac{\varepsilon}{2},$$

whence

$$|x_n - x_{n'}| = |(x_n - a) + (a - x_{n'})|$$

$$\leqslant |x_n - a| + |a - x_{n'}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, the condition of the necessity has been proved. It is more difficult to prove its sufficiency.

Sufficiency. Here we employ the lemma of the preceding subsection.

Thus, assume that the condition is satisfied and according to a given $\varepsilon > 0$ a number N has been found such that for $n > N$ and $n' > N$ inequality (4) is satisfied. Having fixed n', we may rewrite (4) in the form

$$x_{n'} - \varepsilon < x_n < x_{n'} + \varepsilon,$$

so we see that the variable x_n is always bounded; its values for $n > N$ lie between the numbers $x_{n'} - \varepsilon$ and $x_{n'} + \varepsilon$ and it is easy to widen these bounds so that they embrace the first N values x_1, $x_2, \ldots, x_N$.

Now, by the Bolzano–Weierstrass lemma a partial sequence $\{x_{n_k}\}$ can be extracted, which tends to a finite limit c:

$$\lim x_{n_k} = c.$$

We shall prove that this is the limit of the variable x_n itself. k can be selected sufficiently large, so that

$$|x_{n_k} - c| < \varepsilon$$

and at the same time $n_k > N$. Consequently, we may take $n' = n_k$ in (4), i.e.

$$|x_n - x_{n_k}| < \varepsilon.$$

Comparing the two inequalities we finally obtain

$$|x_n - c| < 2\varepsilon \quad (\text{for } n > N),$$

which completes the proof of our assertion†.

Remark. Although Bolzano and Cauchy stated the sufficiency of the above condition of existence of a finite limit, without an exact theory of real numbers it was obviously impossible to prove the statement.

53. The condition of existence of a finite limit for a function of an arbitrary argument. We now proceed to consider the general case of a function $f(x)$ given in a domain $\mathcal{X} = \{x\}$, for which a is a point of condensation. For the existence of a finite limit of this function, when x tends to a, we can establish a criterion similar to that of the case of a function of positive integral argument. The formulation will be given for both a finite a and $a = +\infty$.

THEOREM. *In order that a function $f(x)$ has a finite limit when x tends to a, it is necessary and sufficient that for any number $\varepsilon > 0$ there exists a number $\delta > 0$ ($\Delta > 0$) such that the inequality*

$$|f(x) - f(x')| < \varepsilon$$

is valid, provided that

$$|x - a| < \delta \quad \text{and} \quad |x' - a| < \delta \qquad (x > \Delta \quad \text{and} \quad x' > \Delta).$$

Proof. We carry out the proof assuming that a is a finite number.

Necessity. Let there exist a finite limit

$$\lim_{x \to a} f(x) = A.$$

Then according to a given $\varepsilon > 0$ a number $\delta > 0$ can be found such that

$$|f(x) - A| < \varepsilon/2,$$

if $|x - a| < \delta$. Let also $|x' - a| < \delta$ so that

$$|A - f(x')| < \varepsilon/2.$$

Hence

$$|f(x) - f(x')| < \varepsilon$$

assuming that simultaneously

$$|x - a| < \delta \quad \text{and} \quad |x' - a| < \delta.$$

---

† The number 2ε is to the same extent "an arbitrarily small number" as ε. If it is convenient, we can first take not ε but $\varepsilon/2$ and then we would obtain ε. Similar reasoning we shall, in future, leave to the reader.

Sufficiency. This can, for instance, be established by reducing the case to that already investigated. The proof of this is indicated by the definition of a function "in the language of sequences" [Sec. 32].

Thus, assume that the condition formulated in the theorem is satisfied, and for an arbitrary $\varepsilon > 0$ we have established the corresponding $\delta > 0$.

If $\{x_n\}$ is an arbitrary sequence of values from $\mathfrak{X}$ converging to a, then according to the definition of the limit of a sequence, a number N can be found such that for $n > N$ we have $|x_n - a| < \delta$. Select besides n another number $n' > N$ such that at the same time

$$|x_n - a| < \delta \quad \text{and} \quad |x_{n'} - a| < \delta.$$

Then, by the definition of the number δ,

$$|f(x_n) - f(x_{n'})| < \varepsilon.$$

This inequality is satisfied if both numbers, n and n', are greater than N. This means that for the function $f(x_n)$ of positive integral argument n, the condition of Sec. 52 is satisfied and hence the sequence

$$f(x_1), f(x_2), \ldots, f(x_n), \ldots$$

has a finite limit, say A.

It remains to prove that *this limit A is independent of the selection of the sequence* $\{x_n\}$.

Let $\{x_n'\}$ be another sequence extracted from $\mathfrak{X}$ and also converging to a. The corresponding sequence of values of the function $\{f(x_n')\}$ has, according to what is proved above, a finite limit A'. To prove that $A = A'$, assume the converse. Then we can construct a new sequence

$$x_1, x_1', x_2, x_2', \ldots, x_n, x_n', \ldots$$

of values of x clearly converging to a. It corresponds to the sequence of values of the function

$$f(x_1), f(x_1'), f(x_2), f(x_2'), \ldots, f(x_n), f(x_n'), \ldots,$$

having no limit at all, since the partial sequences of its terms located at odd and even places tend to distinct limits [Sec. 51]. This contradicts the above statement. Thus, when $x \to a$ the function $f(x)$, in fact, tends to a finite limit A.

§ 6. Classification of infinitely small and infinitely large quantities

54. Comparison of infinitesimals. Assume that in some investigation we consider the series of infinitely small quantities

$$\alpha, \beta, \gamma, \ldots$$

which in general are functions of one variable, say x, which tends to a finite or infinite limit a.

In many cases it is of interest to compare the above infinitesimals with respect to their approach to zero. The basis of comparison of infinitesimals α and β is the behaviour of their ratio†. In this connection we establish the following conventions.

I. If the ratio β/α (and so also α/β) has a finite non-vanishing limit, then the infinitesimals α and β are said to be of the same order.

II. If now the ratio β/α itself is an infinitesimal (and the ratio α/β infinitely large), then the infinitesimal β is said to be a quantity of a higher order than the infinitesimal α and also the infinitesimal α of a lower order than β.

For instance, if $\alpha = x \to 0$ then, in comparison with this infinitesimal, the following infinitesimals are of the same order:

$$\sin x, \quad \sqrt{(1+x)} - 1,$$

since we know that [Sec. 34 *(5)*, Sec. 43 *(6)*],

$$\lim_{x \to 0} \frac{\sin x}{x} = 1, \quad \lim_{x \to 0} \frac{\sqrt{(1+x)} - 1}{x} = \frac{1}{2}.$$

However, the infinitesimals

$$1 - \cos x, \quad \tan x - \sin x \tag{1}$$

are evidently of a higher order than x [Sec. 43, *(7)* (a) and (b)].

Of course, it may happen that the ratio of two infinitesimals has no limit at all and is not infinitely large; for instance, if we take [Sec. 34, *(6)* and *(7)*]

$$\alpha = x, \quad \beta = x \sin \frac{1}{x},$$

† We assume that the variable by which we divide is distinct from zero, at least for values of x sufficiently close to a.

their ratio, $\sin(1/x)$, has no limit at all when $x \to 0$. In this case it is said that the two infinitesimals are incomparable.

Observe that if the infinitesimal β is of a higher order than an infinitesimal α, then this is written in the following way:

$$\beta = o(\alpha).$$

For instance, we write

$$1 - \cos x = o(x), \quad \tan x - \sin x = o(x), \text{ etc.}$$

Thus the symbol $o(\alpha)$ is the general notation for an infinitesimal of a higher order than α. This convenient notation will be used in this book.

55. The scale of infinitesimals. Sometimes it is necessary to have a more precise way of comparing the behaviour of infinitesimals; this is done by expressing their ratios by numbers. In this case first we take one of the infinitesimals entering the investigation (say α) as a "standard"; it is called the basic infinitesimal. Evidently, the selection of the basic infinitesimal is to a certain extent arbitrary, but usually the simplest is selected. If, according to our assumption, the considered quantities are functions of x and become infinitely small when x tends to a, then, depending whether a is zero, finite and distinct from zero, or infinite, it is natural to take as the basic infinitesimal

$$|x|, \ |x-a|, \ \frac{1}{|x|}$$

respectively.

Further, we construct for the powers of the basic infinitesimal α (we assume that $\alpha > 0$) with different positive exponents, α^k, a sort of a scale for the estimate of infinitesimals of more complicated nature†.

III. We agree to call the infinitesimal β a quantity of the kth order (with respect to the basic infinitesimal α) if β and α^k $(k > 0)$ are quantities of the same order, i.e. if the ratio β/α^k has a finite non-zero limit.

Now, for instance, being dissatisfied with the statement that the infinitesimals (1) (when $x \to 0$) are quantities of a higher order

† It is readily observed that for $k > 0$ the quantity α^k is infinitesimal together with α.

than $\alpha = x$, we can say that one of them is an infinitesimal of second order, while the second is of third order with respect to $\alpha = x$, since [Sec. 43, *(7)*, (a) and (b)]

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2},\qquad \lim_{x\to 0}\frac{\tan x-\sin x}{x^3}=\frac{1}{2}.$$

56. Equivalent infinitesimals. Consider now the extremely important case of infinitesimals of the same order.

IV. We say that the infinitesimals α and β are *equivalent* (denoted by the symbol $\alpha\sim\beta$) if their difference $\gamma=\beta-\alpha$ is a quantity of a higher order than either of the infinitesimals α and β:

$$\gamma = o(\alpha) \quad \text{and} \quad \gamma = o(\beta).$$

Incidentally it is sufficient to require that γ is of a higher order than one of the above infinitesimals, since if for instance γ is of an order higher than α, then it is of a higher order than β as well. In fact, from the fact that $\lim\gamma/\alpha = 0$ it follows that

$$\lim\frac{\gamma}{\beta}=\lim\frac{\gamma}{\alpha+\gamma}=\lim\frac{\gamma/\alpha}{1+\gamma/\alpha}=0$$

as well.

Consider two equivalent infinitesimals α and β, so that $\beta=\alpha+\gamma$ where $\gamma = o(\alpha)$. If we approximately set $\beta\doteq\alpha$†, then when these quantities decrease, not only the absolute error of this replacement represented by the quantity $|\alpha|$ tends to zero, but also the relative error equal to $|\gamma/\alpha|$ tends to zero. Thus, for sufficiently small values of α and β, we may set $\beta=\alpha$ with an arbitrarily large relative accuracy. This is the basis of replacing, in approximate calculations, complicated infinitesimals by simpler ones.

We now establish a useful criterion on the equivalency of two infinitesimals, which in fact constitutes a second (equivalent) definition of this concept:

In order that the two infinitesimals α and β are equivalent, it is necessary and sufficient that

$$\lim\frac{\beta}{\alpha}=1.$$

† The sign $\doteq$ means "approximately equal to."

Setting $\beta - \alpha = \gamma$ we have

$$\frac{\beta}{\alpha} - 1 = \frac{\gamma}{\alpha}.$$

This at once implies our assertion. In fact, if $\beta/\alpha \to 1$ then $\gamma/\alpha \to 0$, i.e. γ is an infinitesimal of a higher order than α and $\beta \sim \alpha$. Conversely, if we know that $\beta \sim \alpha$, then $\gamma/\alpha \to 0$ and hence $\beta/\alpha \to 1$.

This criterion for instance implies that when $x \to 0$ the infinitesimal $\sin x$ is equivalent to x and $\sqrt{(1+x)} - 1$ to $x/2$. Hence we have the approximate formulae

$$\sin x \doteq x, \quad \sqrt{(1+x)} - 1 \doteq \tfrac{1}{2}x.$$

This property of equivalent infinitesimals leads to their use in solving indeterminancies of the type 0/0, i.e. in finding the limit of the ratio of two infinitesimals β/α. Each of them can be replaced without affecting the limit by any equivalent infinitesimal.

In fact, if $\bar{\alpha} \sim \alpha$ and $\bar{\beta} \sim \beta$, i.e.

$$\lim \frac{\bar{\alpha}}{\alpha} = 1 \quad \text{and} \quad \lim \frac{\beta}{\bar{\beta}} = 1,$$

he ratio

$$\frac{\beta}{\alpha} = \frac{\beta}{\bar{\beta}} \cdot \frac{\bar{\beta}}{\bar{\alpha}} \cdot \frac{\bar{\alpha}}{\alpha},$$

differing from the ratio $\bar{\beta}/\bar{\alpha}$ by factors tending to unity, has the same limits as $\bar{\beta}/\bar{\alpha}$.

We can often simplify the problem by selecting suitable $\bar{\alpha}$ and $\bar{\beta}$, for instance

$$\lim_{x \to 0} \frac{\sqrt{(1+x+x^2)} - 1}{\sin 2x} = \lim_{x \to 0} \frac{\frac{1}{2}(x+x^2)}{2x} = \frac{1}{4}.$$

The above result also implies that two infinitesimals which are in turn equivalent to a third one are mutually equivalent.

57. Separation of the principal part. If the basic infinitesimal α is selected the simplest infinitesimals are quantities of the form $c \cdot \alpha^k$ where c is a constant coefficient and $k > 0$. Let the infinitesimal β be of kth order with respect to α, i.e.

$$\lim \frac{\beta}{\alpha^k} = c,$$

c being a finite non-zero number. Then

$$\lim \frac{\beta}{c\alpha^k} = 1,$$

and so the infinitesimals β and $c\alpha^k$ are equivalent: $\beta \sim c\alpha^k$. The infinitesimal $c\alpha^k$, equivalent to the given infinitesimal β, is called its *principal part* (or *principal term*).

Making use of the results proved above, together with the simple examples, it is easy to separate the principal parts of the expressions

$$1 - \cos x \sim \tfrac{1}{2}x^2, \quad \tan x - \sin x \sim \tfrac{1}{2}x^3.$$

Here $x \to 0$ and $\alpha = x$ is the basic infinitesimal.

Let $\beta \sim c\alpha^k$, i.e. $\beta = c\alpha^k + \gamma$ where $\gamma = o(\alpha^k)$. We can imagine that we have again separated the principal term of the infinitesimal γ: $\gamma = c'\alpha^{k'} + \delta$ where $\delta = o(\alpha^{k'})$ $(k' > k)$, etc.

This process of successive separation from an infinitesimal of its simplest infinitesimals of increasing orders can be repeated.

We confine ourselves in this section to establishing general concepts, illustrating these by a few examples. In what follows, we shall give a systematic device, both for constructing the principal part of a given infinitesimal and for the further separation from it of the simplest infinitesimals.

58. Problems. To illustrate the above results we give two problems in which they are used.

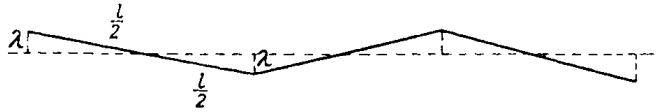

FIG. 25.

(*1*) Suppose that the length of a straight line lying in a given plane is measured by means of a ruler l metres in length. If the ruler is applied not exactly along the straight line to be measured, the result of the measurement will turn out to be somewhat greater than the real length. We assume that the ruler is applied in a zigzag fashion so that its ends are each a distance λ metres from the straight line on alternate sides (Fig. 25). It is required to estimate the error.

In applying the ruler, the error each time is equal to the difference between the length l of the ruler and its projection on the measured curve; the considered projection is

$$2\sqrt{\left[\left(\frac{l}{2}\right)^2 - \lambda^2\right]} = l\sqrt{\left[1 - \frac{4\lambda^2}{l^2}\right]}.$$

Making use of the approximate formula

$$\sqrt{(1+x)} \doteq 1 + \tfrac{1}{2}x$$

for $x = -4\lambda^2/l^2$ (this is justified by the smallness of the quantity λ as compared with l) we replace the expression for the projection by

$$l\left(1 - \frac{2\lambda^2}{l^2}\right) = l - \frac{2\lambda^2}{l}.$$

In this case the above error is $2\lambda^2/l$ and the relative error is clearly $2\lambda^2/l^2$. The same relative error occurs in a repeated application of the ruler.

If for this error the bound δ is established, i.e. we should have $2\lambda^2/l^2 < \delta$, this implies that $\lambda < l\sqrt{(\delta/2)}$.

For instance, when measuring with a two-metre ruler ($l = 2$), to obtain the relative accuracy of 0.001 it is necessary that the deviation λ be not greater than $2\sqrt{0.0005} \doteq 0.045$ m, i.e. 4.5 cm.

(2) When subdividing a circle into arcs it is often of interest to find the ratio of the height $f = DB$ of the arc ABC of the circle to the height $f_1 = D_1B_1$ of a half AB_1B of this arc (Fig. 26).

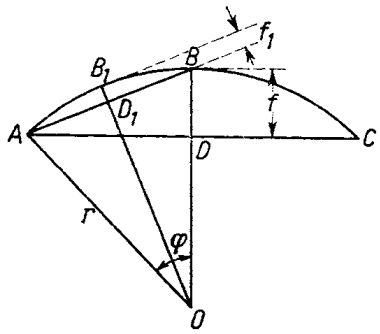

FIG. 26.

If the radius of the circle is r, then

$$\sphericalangle AOB = \varphi, \quad \text{so} \quad \sphericalangle AOB_1 = \tfrac{1}{2}\varphi$$

and

$$f = DB = r(1 - \cos\varphi), f_1 = r(1 - \cos\tfrac{1}{2}\varphi).$$

Thus, the required ratio is equal to

$$\frac{f}{f_1} = \frac{1 - \cos\varphi}{1 - \cos\frac{1}{2}\varphi}.$$

This expression is too complicated to be conveniently used in practice. Let us find its limit when $\varphi \to 0$ (since for sufficiently small φ this expression can approximately be replaced by its limit). For this purpose let us replace the numerator and the denominator by their principal parts. Then we obtain at once

$$\lim\frac{f}{f_1} = \lim\frac{\frac{1}{2}\varphi^2}{\frac{1}{2}(\frac{1}{2}\varphi)^2} = 4.$$

Thus, for arcs corresponding to a small central angle we may approximately assume that the height of half of an arc is four times smaller than the height of the arc itself. This makes it possible to construct, approximately, the arc the ends and the centre of which are given.

59. Classification of infinitely large quantities. We see that for infinitely large quantities a similar classification can be developed. As in Sec. 54, we regard the considered quantities as functions of one variable x, the functions becoming infinitely large when x tends to a.

I. Two infinitely large quantities y and z are said to be of the same order if their ratio z/y (and so also y/z) has a finite and non-zero limit.

II. If now the ratio z/y itself is infinitely large (and the inverse ratio infinitely small), then z is regarded as an infinitely large quantity of a *higher order* than y and y is of a *lower order* than z.

In the case when the ratio z/y does not tend to any finite limit at all but is not infinitely large, then the infinitely large quantities y and z are said to be *incomparable.*

In simultaneous consideration of a number of infinitely large quantities one of them (say y) can be selected to be the basic one and its powers are compared with the remaining infinitely large quantities. For instance, if (as we assumed before) they are all functions of x and become infinitely large when $x \to a$ then, for the basic infinitely large quantity, we usually select $|x|$ if $a = \pm\infty$ and $1/|x-a|$ if a is finite.

III. An infinitely large z is said to be a quantity of the kth order (with respect to the basic infinitely large y) if z and y^k are of the same order, i.e. if the ratio z/y^k has a finite non-zero limit.

CHAPTER 4

CONTINUOUS FUNCTIONS OF ONE VARIABLE

§ 1. Continuity (and discontinuity) of a function

60. Definition of the continuity of a function at a point. The concept of the limit of a function is closely related to another important concept of mathematical analysis, namely the concept of continuity of a function. The creation of this concept in a precise form is due to Bolzano and Cauchy, whose names have already been mentioned.

Consider a function $f(x)$ defined in an interval $\mathcal{X}$, and let x_0 be a point of this interval, so that at this point the function has a definite value $f(x_0)$.

In establishing the concept of limit of a function when x tends to x_0 [Secs. 32, 33]

$$\lim_{x \to x_0} f(x),$$

we frequently emphasized that the value x_0 is never taken by the variable x; moreover, this value may not belong to the domain of definition of the function, and if it does belong to it, when constructing the considered limit the value $f(x_0)$ was not taken into account.

However, the case when

$$\lim_{x \to x_0} f(x) = f(x_0) \tag{1}$$

is of special interest. It is said that the function $f(x)$ is continuous at the value $x = x_0$ (i.e. at the point $x = x_0$) if the last relation is valid; if it is not satisfied, it is said that at this value (or at this point) the function has a *discontinuity*†.

† This terminology is connected with the *intuitive* representation of continuity and discontinuities of a curve; the function is continuous if its graph

In the case of the continuity of the function $f(x)$ at the point x_0 (and evidently only in this case), in computing the limit of the function $f(x)$ when $x \to x_0$, it is irrelevant whether x tending to x_0 takes the particular value x_0 or not.

The definition of continuity of a function may also be formulated in another way. Passing from the value x_0 to another value x can be performed by adding to the value x_0 an increment $\Delta x = x - x_0$†. The new value of the function $y = f(x) = f(x_0 + \Delta x)$ differs from the old value $y_0 = f(x_0)$ by the increment

$$\Delta y = f(x) - f(x_0) = f(x_0 + \Delta x) - f(x_0).$$

In order that the function $f(x)$ be continuous at the point x_0, it is necessary and sufficient that its increment Δy at this point tends to zero when the increment Δx of the independent variable tends to zero. In other words, continuous functions are determined by the property that to an infinitesimal increment of the argument there corresponds an infinitesimal increment of the function.

Returning to the basic definition (1), let us find its meaning "in the language of sequences" [Sec. 32]. The concept of continuity of a function $f(x)$ at the point x_0 can be defined as follows: for any sequence of values of x from $\mathcal{X}$

$$x_1, x_2, \ldots, x_n, \ldots$$

converging to x_0, the corresponding sequence of values of the function

$$f(x_1), f(x_2), \ldots, f(x_n), \ldots$$

converges to $f(x_0)$.

Finally, "in the ε–δ language" [Sec. 33] the continuity is expressed as follows: for an arbitrary number $\varepsilon > 0$ a number $\delta > 0$ can be found for it, such that

$$|x - x_0| < \delta \quad \text{implies} \quad |f(x) - f(x_0)| < \varepsilon.$$

---

is continuous, the points of discontinuity of a function correspond to the points of discontinuity of its graph. However, in fact, the concept of continuity for a curve also requires a justification and the simplest way to provide it is to use the continuity of a function.

† In analysis it is customary to denote the increments of the quantities $x, y, t, \ldots$ by $\Delta x, \Delta y, \Delta t, \ldots$. These notations are to be regarded as complete symbols, i.e. we cannot separate Δ from x, etc.

Thus, the last inequality should be satisfied in a sufficiently small neighbourhood $(x_0 - \delta, x_0 + \delta)$ of the point x_0.

Observe that computing the limit (1) we can approach x_0 from the left and from the right, provided that x does not take values outside the interval $\mathfrak{X}$.

We now proceed to establish the concept of *one-sided continuity* or *one-sided discontinuity* of a function at a point.

It is said that the function $f(x)$ is continuous at point x_0 from the right (from the left) if the limit relation

$$\left.\begin{aligned} & f(x_0 + 0) = \lim_{x \to x_0 + 0} f(x) = f(x_0) \\ [\text{or} \quad & f(x_0 - 0) = \lim_{x \to x_0 - 0} f(x) = f(x_0)] \end{aligned}\right\} \tag{2}$$

is satisfied. If one of these relations is not satisfied, then function $f(x)$ has at point x_0 a discontinuity from the right or from the left, respectively.

At the left-(right-)hand end of the interval $\mathfrak{X}$† in which the function is defined, it is evident that we can only consider continuity or discontinuity from the right (from the left). If, however, x is an interior point of the interval $\mathfrak{X}$, i.e. it does not coincide with one of the end-points, then in order that relation (1) (which expresses the continuity of the function at point x_0 in the ordinary sense) be satisfied, it is necessary and sufficient that both of the relations of (2) are satisfied *simultaneously* [Sec. 35]. In other words, the continuity of a function at the point x_0 is equivalent to its continuity at this point simultaneously from the right and from the left.

To simplify description of a function we say that it is continuous in the interval $\mathfrak{X}$ if it is continuous at every point of this interval.

61. Condition of continuity of a monotonic function. Consider the function $f(x)$ which increases (decreases) monotonically‡ in the interval $\mathfrak{X}$ [Sec. 47]. This interval can be finite or infinite, closed, semi-open or open. We now establish a simple criterion which

† Assuming that this end-point is a finite number.

‡ For clarity we assume that the function is monotonically increasing in the strict sense (although the theorem is also valid for monotonic functions in the wider sense).

enables us to determine whether functions of this type are continuous over the whole interval $\mathcal{X}$.

THEOREM. *If the set of values of a monotonically increasing (decreasing) function $f(x)$ which it takes when x ranges over the interval $\mathcal{X}$ is contained in an interval $\mathcal{Y}$ and fills the latter continuously, then the function $f(x)$ is continuous in the interval $\mathcal{X}$*†.

Take an arbitrary point x_0 in $\mathcal{X}$; assuming that it is not the right-hand end of this interval we shall prove that the function $f(x)$ is continuous at the point x_0 from the right; similarly, the continuity of the function can be proved at point x_0 from the left, if x_0 is not the left-hand end of the considered interval; from these two results the proof of the theorem readily follows.

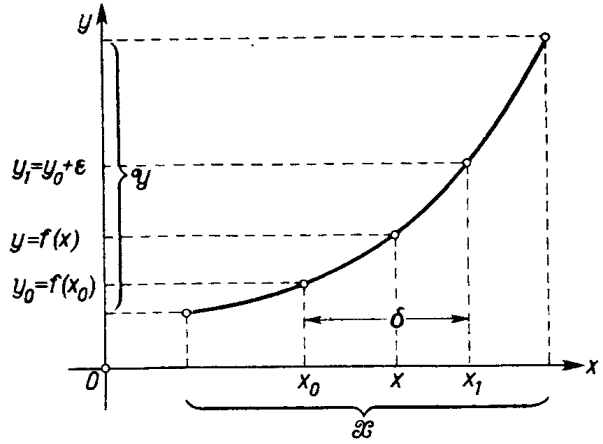

FIG. 27.

The point $y_0 = f(x_0)$ belongs to interval $\mathcal{Y}$ and is not its right-hand end (since in $\mathcal{X}$ there are values $x > x_0$ and to them there correspond in $\mathcal{Y}$ the values $y = f(x) > y_0$). Let ε be an arbitrary small positive number; in fact, we assume it to be so small that also the value $y_1 = y_0 + \varepsilon$ belongs to interval $\mathcal{Y}$. Since, by assumption, $\mathcal{Y} = \{f(x)\}$, then in $\mathcal{X}$ a value x_1 can be found such that

$$f(x_1) = y_1,$$

† Subsequently [Sec. 70] we shall prove that the condition which is formulated here as sufficient for the continuity of a monotonic function is also necessary.

and it is evident that $x_1 > x_0$ (since for $x \leqslant x_0$, $f(x) \leqslant y_0$). Set $\delta = x_1 - x_0$ so that $x_1 = x_0 + \delta$. If now

$$0 < x - x_0 < \delta, \quad \text{i.e.} \quad x_0 < x < x_1,$$

then

$$y_0 < f(x) < y_1 = y_0 + \varepsilon \quad \text{or} \quad 0 < f(x) - f(x_0) < \varepsilon.$$

This implies that

$$\lim_{x \to x_0 + 0} f(x) = f(x_0),$$

i.e. function $f(x)$ is in fact continuous at the point x_0 from the right. This completes the proof. Figure 27 presents our reasoning diagrammatically.

62. Arithmetical operations over continuous functions. Before proceeding to examples of continuous functions we establish a simple proposition which enables us to increase their number.

THEOREM. *If two functions $f(x)$ and $g(x)$ are defined in one interval $\mathcal{X}$ and both are continuous at a point x_0, then the functions*

$$f(x) \pm g(x), \quad f(x) \cdot g(x), \quad \frac{f(x)}{g(x)}$$

are also continuous at this point (the last under the condition that $g(x_0) \neq 0$).

The theorem follows directly from theorems on the limit of a sum, difference, product and quotient of two functions, each having a limit [Sec. 42].

Consider as an example the quotient of two functions. The assumption of the continuity of the functions $f(x)$ and $g(x)$ at point x_0 is equivalent to the two relations

$$\lim_{x \to x_0} f(x) = f(x_0), \quad \lim_{x \to x_0} g(x) = g(x_0).$$

But, according to the theorem on the limit of a quotient, it follows (since the limit of the denominator is not zero) that

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f(x_0)}{g(x_0)}.$$

This relation implies that function $f(x)/g(x)$ is continuous at the point x_0.

63. Continuity of elementary functions. *(1) Integral and fractional rational function.* The continuity of functions of x reducing to a

constant or x itself is obvious. Hence, in view of the theorem of the preceding subsection, we infer the continuity of any expression

$$ax^m = a \cdot \overbrace{x \cdot x \dots x}^{m \text{ times}}$$

having one term as the product of continuous functions, and moreover, the continuity of a polynomial (integral rational function)

$$a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n$$

as a sum of continuous functions. In all the above cases the continuity occurs in the whole interval $(-\infty, +\infty)$.

Finally, it is evident that also the quotient of two polynomials (a fractional rational function)

$$\frac{a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n}{b_0 x^m + b_1 x^{m-1} + \dots + b_{m-1} x + b_m}$$

is continuous for any value of x, except those at which the denominator is zero.

The continuity of the other elementary functions will be established using the theorem of Sec. 61.

(2) Exponential function, $y = a^x (a > 1)$. This is monotonically increasing for x increasing in the interval $\mathcal{X} = (-\infty, +\infty)$. Its values are positive and fill the whole interval $\mathcal{Y} = (0, +\infty)$; this is seen from the existence of the logarithm $x = \log_a y$ for any $y > 0$ [Sec. 12]. Consequently, the exponential function is continuous for all values of x.

(3) Logarithmic function $y = \log_a x$ $(a > 0, a \neq 1)$. Confining ourselves to the case $a > 1$ we observe that this function increases when x increases the interval $\mathcal{X} = (0, +\infty)$. Moreover, it evidently takes every value of y from the interval $\mathcal{Y} = (-\infty, +\infty)$, namely for $x = a^y$. This implies its continuity.

(4) Power function $y = x^\mu (\mu \gtrless 0)$. When x increases from zero to $+\infty$ it increases if $\mu > 0$, and decreases if $\mu < 0$. It takes all positive values of y (for $x = y^{1/\mu}$); consequently, it is continuous†.

† If $\mu > 0$ the value zero is included into both intervals of variability of x and y; when $\mu < 0$ the zero is not included. Further, if μ is an integer $\pm n$ or a fraction $\pm p/q$ with an odd denominator, then the power x^μ can be considered also for $x < 0$; its continuity for these values is established in an analogous way.

(5) Trigonometric functions.

$$y = \sin x,\ y = \cos x,\ y = \tan x,\ y = \cot x,$$
$$y = \sec x,\ y = \operatorname{cosec} x.$$

Consider first the function $y = \sin x$. Its continuity, say for x ranging over the interval $\mathcal{X} = [-\pi/2, +\pi/2]$, follows from its monotonicity in this interval and from the fact (established geometrically) that it then takes all values between -1 and $+1$. The same is true for any interval of the form

$$\left[k\pi - \frac{\pi}{2},\ k\pi + \frac{\pi}{2}\right] \quad (k = 0, \pm 1, \pm 2, \ldots).$$

We finally observe that the function $y = \sin x$ is continuous for all values of x. Similarly we can establish the continuity of the function $y = \cos x$ for an arbitrary value of x.

Hence, by the theorem of the preceding subsection, this implies the continuity of the functions

$$\tan x = \frac{\sin x}{\cos x},\ \sec x = \frac{1}{\cos x},\ \cot x = \frac{\cos x}{\sin x},\ \operatorname{cosec} x = \frac{1}{\sin x}.$$

An exception arises for the first two functions for the values $(2k+1)\pi/2$ for which $\cos x$ vanishes, and for the last two functions for the values $k\pi$ for which $\sin x$ vanishes.

Finally let us examine

(6) Inverse trigonometric functions.

$$y = \operatorname{arc\,sin} x,\ y = \operatorname{arc\,cos} x,\ y = \operatorname{arc\,tan} x,\ y = \operatorname{arc\,cot} x.$$

The first two are continuous in the interval $[-1, +1]$ and the last two in the interval $(-\infty, +\infty)$. The proof is left to the reader.

Summing up, we may say that the basic elementary functions are continuous at all points where they have meaning, i.e. in the corresponding natural domains of their definition.

64. The superposition of continuous functions. Wide classes of continuous functions can be constructed by means of superposition applied to functions the continuity of which has already been established. The basis is constituted by the following

THEOREM. *Let the function $\varphi(y)$ be defined in the interval $\mathcal{Y}$ and the function $f(x)$ in the interval $\mathcal{X}$, the values of the latter function*

remaining within the bounds of $\mathcal{Y}$ for x in $\mathcal{X}$. If $f(x)$ is continuous at the point x_0 in $\mathcal{X}$, and $\varphi(y)$ is continuous at the corresponding point $y_0 = f(x_0)$ in $\mathcal{Y}$, then the compound function $\varphi(f(x))$ is also continuous at the point x_0.

Proof. Take an arbitrary number $\varepsilon > 0$. Since $\varphi(y)$ is continuous at $y = y_0$, we can find $\sigma > 0$ (depending on ε) such that

$$|y - y_0| < \sigma \quad \text{implies} \quad |\varphi(y) - \varphi(y_0)| < \varepsilon.$$

On the other hand, in view of the continuity of $f(x)$ for $x = x_0$, we can find $\delta > 0$ (depending on σ) such that

$$|x - x_0| < \delta \quad \text{implies} \quad |f(x) - f(x_0)| = |f(x) - y_0| < \sigma.$$

From the very choice of number σ it follows that

$$|\varphi(f(x)) - \varphi(y_0)| = |\varphi(f(x)) - \varphi(f(x_0))| < \varepsilon.$$

This proves the continuity of the function $\varphi(f(x))$ at point x_0 "in the ε-δ language".

For instance, if the power function x^μ $(x > 0)$ is represented in the form

$$x^\mu = e^{\mu \log x}$$

obtained by superposition of the logarithmic and exponential functions, then from the continuity of the latter two functions we can deduce the continuity of the power function.

65. Computation of certain limits. The continuity of functions can be used in numerous ways in computing limits†. Here, using the continuity of elementary functions, we shall establish a number of important limits which will be required in the next chapter:

(1) $\lim\limits_{\alpha \to 0} \dfrac{\log_a(1+\alpha)}{\alpha} = \log_a e \quad \left(\dfrac{0}{0}\right)$,

(2) $\lim\limits_{\alpha \to 0} \dfrac{a^\alpha - 1}{\alpha} = \log a \quad \left(\dfrac{0}{0}\right)$,

(3) $\lim\limits_{\alpha \to 0} \dfrac{(1+\alpha)^\mu - 1}{\alpha} = \mu \quad \left(\dfrac{0}{0}\right)$.

† Actually we have done so before, e.g. in Example *(6)* of Sec. 43 we incidentally established the continuity of $\sqrt[n]{x}$ for $x = 1$ and then employed it; in Example *(7)* (b) we did the same for the function $\cos x$ for $x = 0$.

We have

$$\frac{\log_a(1+\alpha)}{\alpha} = \log_a(1+\alpha)^{\frac{1}{\alpha}};$$

since the expression on the right under the logarithm sign tends to e when $\alpha \to 0$ [Sec. 50, (4)], then (by the continuity of the logarithmic function) its logarithm tends to $\log_a e$; this completes the proof.

We should note a particular example of the formula proved above; when the natural logarithm is considered ($a = e$)

$$\lim_{\alpha \to 0} \frac{\log(1+\alpha)}{\alpha} = 1.$$

The simplicity of this result is the main reason for the advantages of the natural system of logarithms.

Now, set in formula *(2)* $a^\alpha - 1 = \beta$; then for $\alpha \to 0$ (by the continuity of the exponential function), $\beta \to 0$. Further we have $\alpha = \log_a(1+\beta)$ and hence, making use of the above result,

$$\lim_{\alpha \to 0} \frac{a^\alpha - 1}{\alpha} = \lim_{\beta \to 0} \frac{\beta}{\log_a(1+\beta)} = \frac{1}{\log_a e} = \log a.$$

This completes the proof.

If we take in particular $\alpha = 1/n$ ($n = 1, 2, 3, \ldots$) we arrive at an interesting formula

$$\lim_{n \to \infty} n(\sqrt[n]{a} - 1) = \log a \qquad (\infty \cdot 0).$$

Finally, to prove formula *(3)* we set $(1+\alpha)^\mu - 1 = \beta$; for $\alpha \to 0$ (by the continuity of a power function) we obtain also $\beta \to 0$. Taking logarithms in the expression $(1+\alpha)^\mu = 1 + \beta$ we have

$$\mu \cdot \log(1+\alpha) = \log(1+\beta).$$

By means of this relation we transform the considered expression as follows:

$$\frac{(1+\alpha)^\mu - 1}{\alpha} = \frac{\beta}{\alpha} = \frac{\beta}{\log(1+\beta)} \cdot \mu \cdot \frac{\log(1+\alpha)}{\alpha}.$$

The two ratios

$$\frac{\beta}{\log(1+\beta)} \quad \text{and} \quad \frac{\log(1+\alpha)}{\alpha}$$

tend to unity and hence the whole product has the limit μ. This completes the proof.

The limit examined in Sec. 43, *(6)* is the particular case of $\mu = 1/2$.

66. Power-exponential expressions. Consider now the *power-exponential* expression u^v where u and v are functions of one variable x with the domain of variation $\mathfrak{X}$ having the point of condensation x_0; in particular, we can have two functions u_n and v_n of a positive integral argument.

Let there exist the finite limits

$$\lim_{x \to x_0} u = a \quad \text{and} \quad \lim_{x \to x} v = b,$$

where $a > 0$. It is required to find the limit of the expression u^v.

Represent it in the form

$$u^v = e^{v \cdot \log u};$$

the functions v and $\log u$ have the limits

$$\lim_{x \to x_0} v = b, \quad \lim_{x \to x_0} \log u = \log a$$

(we have used the continuity of a logarithmic function), so that

$$\lim_{x \to x_0} v \cdot \log u = b \cdot \log a.$$

Hence, by the continuity of an exponential function, we finally have

$$\lim_{x \to x_0} u^v = e^{b \cdot \log a} = a^b.$$

The limit of the expression u^v can also be established in other cases—when the limit c of the product $v \cdot \log u$ is known, no matter whether it is finite or not. For a finite c it is evident that the required limit is e^c; if, however, $c = -\infty$ or $+\infty$, the limit is 0 or $+\infty$, respectively [Sec. 34, *(2)*].

The determination of the limit $c = \lim\{v \cdot \log u\}$ when only the limits a and b are prescribed is always possible except in the cases when this product represents an indeterminancy of the type $0 \cdot \infty$ when $x \to x_0$. It is readily observed that the exceptional cases correspond to the following combinations of values a and b:

$$a = 1, \qquad b = \pm\infty;$$
$$a = 0, \qquad b = 0;$$
$$a = +\infty, \qquad b = 0.$$

In these cases it is said that the expression u^v represents an indeterminate form of the type 1^∞, 0^0, ∞^0† (depending on the case). To solve the problem of the limit of the expression u^v it is insufficient to know only the limits of the functions u and v; it is necessary to take into account the way in which they tend to their limits.

The expression $(1+1/n)^n$ when $n \to \infty$, or the more general expression $(1+\alpha)^{1/\alpha}$ which has the limit e for $\alpha \to 0$, are examples of indeterminate forms of the type 1^∞.

It has already been stated that general methods of solving indeterminate forms will be dealt with in § 3 of Chapter 7.

67. Classification of discontinuities. Examples. Let us consider in more detail the problem of the continuity and dicontinuities of functions at a point x, say from the right. Assuming that the function is defined in a certain interval $[x_0, x_0+h]$ $(h>0)$ on the right from this point, we observe that for the continuity it is necessary and sufficient that, first, there exists a finite limit $f(x_0+0)$ of the function $f(x)$ when x tends to x_0 from the right, and, secondly, that this limit is equal to the value of the function $f(x_0)$ at the point x_0.

Hence, it is easy to see when a discontinuity on the right of the function $f(x)$ at the point x_0 may occur. It may happen that although a finite limit $f(x_0+0)$ exists, it is not equal to the value $f(x_0)$; such discontinuity is called *ordinary* or a discontinuity of the first kind‡. Now it may also happen that the limit $f(x_0+0)$ is infinite or it does not exist at all; then we say that there is a discontinuity of the second kind.

If the function $f(x)$ is defined only in the interval $(x_0, x_0+h]$, but the finite limit exists, then it is only necessary to define the function at the point x_0,

$$f(x_0+0) = \lim_{x \to x_0+0} f(x),$$

setting $f(x_0)$ equal to the limit, in order that the function be continuous from the right at the point x_0. This will hereafter usually be implicitly assumed. Incidentally, if the function is also defined on the left from x_0, i.e. in the interval $[x_0-h, x_0]$, and the finite limit

$$f(x_0-0) = \lim_{x \to x_0-0} f(x),$$

exists, then continuity of the function at point x_0 can only be achieved if the two limits are equal.

Finally, if there does not exist a limit for the function $f(x)$ defined in the interval $(x_0, x_0+h]$, at a point x_0, then it is said that the function has at point x_0 a discon-

† Concerning these symbols, we repeat our comment in the footnote on p. 83.

‡ It is also said in this case that the function $f(x)$ has a jump from the right at point x_0, the magnitude of which is equal to $f(x_0+0)-f(x_0)$.

tinuity of the second kind from the right, regardless of the fact that it is not defined at all at this point; in this case, no matter how we define the function at x_0, it necessarily has a discontinuity.

Examples. (1) Consider the function $y = E(x)$ (its graph is given in Fig. 5). If x_0 is not an integer and $E(x_0) = m$, i.e. $m < x_0 < m+1$, then for all values of x in the interval $(m, m+1)$ we have $E(x) = m$, and hence the continuity of the function at the point x_0 is obvious.

The case is different if x_0 is equal to an integer m. Continuity occurs on the right at this point, since on the right of $x_0 = m$, for the values of x in $(m, m+1)$, we have $E(x) = m$, and hence also $E(m+0) = m = E(m)$. However, on the left of $x_0 = m$, for values of x in $(m-1, m)$ it is evident that $E(x) = m-1$; hence $E(m-0) = m-1$, which is not equal to the value of $E(m)$, and at the point $x_0 = m$ the function has from the left an ordinary discontinuity or a jump

(2) For the function

$$f(x) = \frac{1}{x^3} \quad (\text{for } x \neq 0)$$

the point $x = 0$ is a point of discontinuity of the second kind from both sides; at this point the function tends to infinity from the left and from the right:

$$f(+0) = \lim_{x \to +0} \frac{1}{x^3} = +\infty, \quad f(-0) = \lim_{x \to -0} \frac{1}{x^3} = -\infty.$$

(3) The function

$$f(x) = \sin \frac{1}{x} \quad (\text{for } x \neq 0),$$

considered in Sec. 34, *(6)*, has at the point $x = 0$ a discontinuity of the second kind from both sides, since there does not exist any limit of this function at the considered point, either from the right or from the left.

(4) Now, if we take the function [Sec. 34, *(7)*]

$$f(x) = x \sin \frac{1}{x} \quad (\text{for } x \neq 0)$$

for which, as we have already found, there exists the limit

$$\lim_{x \to 0} f(x) = 0,$$

then setting $f(0) = 0$ we re-establish continuity for $x = 0$.

(5) Let us finally define two functions by the relations

$$f_1(x) = a^{\frac{1}{x}} \ (a > 1), \quad f_2(x) = \operatorname{arc\,tan} \frac{1}{x}$$

for $x \neq 0$, and by the additional conditions

$$f_1(0) = f_2(0) = 0.$$

We have already seen [Sec. 35] that

$$f_1(+0) = +\infty, \quad f_1(-0) = 0,$$

$$f_2(+0) = \frac{\pi}{2}, \qquad f_2(-0) = -\frac{\pi}{2}.$$

Thus, at the point $x = 0$ the first function has a discontinuity of the second kind from the right, and the second function has jumps from both sides (cf. Figs. 22 and 23).

To conclude the section we consider an important class of frequently encountered functions—the monotonic or piecewise monotonic functions†. We shall prove that in this case only ordinary discontinuities may occur. This follows from the fact that for such a function $f(x)$, at all points x of the interval $\mathfrak{X}$ where this function is defined, there always exist finite limits $f(x_0+0)$ and $f(x_0-0)$ (or just one of them if x_0 is an end of the interval $\mathfrak{X}$). Suppose, for instance, that $f(x)$ increases monotonically and x_0 is the left end of the interval $\mathfrak{X}$; then for $x < x_0$, the values $f(x)$ are bounded from above by the number $f(x_0)$ and according to the theorem of Sec. 47 there exists a finite limit

$$f(x_0-0) = \lim_{x \to x_0-0} f(x).$$

§ 2. Properties of continuous functions

68. Theorem on the zeros of a function. We now investigate the basic properties of functions continuous in a certain interval. These properties are interesting and form the basis of various propositions.

The first person to establish strict foundations for the above properties was Bolzano (1817), followed by Cauchy (1821). We owe to them the following.

FIRST BOLZANO–CAUCHY THEOREM. *Suppose that a function $f(x)$ is defined and continuous in a closed interval $[a, b]$ and on the ends of this interval its values have opposite signs. Then between a and b there always exists a point c at which the function has a zero*

$$f(c) = 0 \quad (a < c < b).$$

The theorem has a very simple geometric interpretation: if a continuous curve passes from one side of the x-axis on the other side, then it intersects this axis (Fig. 28).

† A function is called *piecewise-monotonic* if the interval of its definition can be divided into a finite number of partial intervals, in each of which the function is monotonic.

Proof. This will be carried out by the method of subdivision of the interval [Sec. 51]. For definiteness, assume that $f(a) < 0$ and $f(b) > 0$. Divide the interval into halves by the point $(a+b)/2$. It may happen that the function $f(x)$ vanishes at this point, and then the theorem is proved, since we can set $c = (a+b)/2$. Therefore let $f((a+b)/2) \neq 0$; then at the ends of one of the intervals

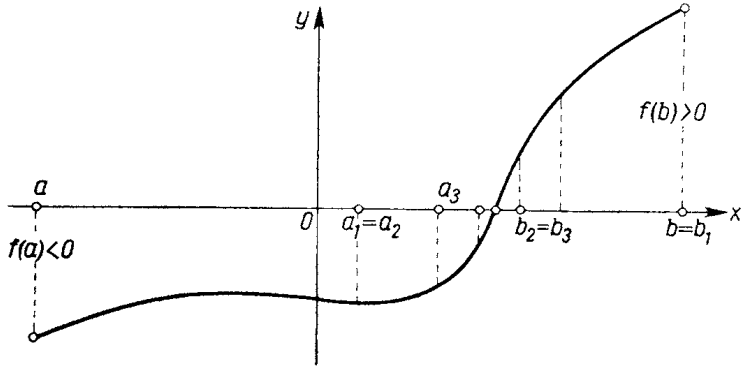

FIG. 28.

$[a, (a+b)/2]$, $[(a+b)/2, b]$ the function takes values having different signs (and moreover, the negative sign at the left end and the positive at the right end). Denoting this interval by $[a_1, b_1]$ we obtain

$$f(a_1) < 0, \quad f(b_1) > 0.$$

Now divide the interval $[a_1, b_1]$ into halves and again disregard the case when $f(x)$ vanishes at the centre $(a_1+b_1)/2$ of this interval, since then the theorem would be proved. Denote by $[a_2, b_2]$ the half for which

$$f(a_2) < 0, \quad f(b_2) > 0.$$

We continue the process of construction of the intervals. Then, after a finite number of steps, either we obtain as the point of division a point at which the function vanishes—and this completes the proof—or we obtain an infinite sequence of imbedded intervals. In the latter case for the nth interval $[a_n, b_n]$, $(n = 1, 2, 3, \ldots)$ we have

$$f(a_n) < 0, \quad f(b_n) > 0, \tag{1}$$

and evidently its length is

$$b_n - a_n = \frac{b-a}{2^n}. \tag{2}$$

The constructed sequence of intervals satisfies the conditions of the lemma on imbedded intervals [Sec. 46] since, in view of (2), $\lim(b_n - a_n) = 0$; hence both variables a_n and b_n tend to the common limit

$$\lim a_n = \lim b_n = c,$$

which evidently belongs to $[a, b]$ [Sec. 36, *(3)*]. We shall prove that this point c satisfies the conditions of the theorem.

Passing to the limit in inequalities (1) and making use of the continuity of the function (in particular at point $x = c$), we find that, simultaneously,

$$f(c) = \lim f(a_n) \leqslant 0 \quad \text{and} \quad f(c) = \lim f(b_n) \geqslant 0,$$

and hence, in fact, $f(c) = 0$. This completes the proof.

Observe that the requirement of continuity of the function $f(x)$ in the closed interval $[a, b]$ is essential; a function having a discontinuity at even one point can pass from a negative value to a positive one without vanishing. For instance, this is the case for $f(x) = E(x) - 1/2$, which does not vanish anywhere although $f(0) = -1/2$ and $f(1) = 1/2$ (there is a jump at $x = 1$).

69. Application to the solution of equations. The above theorem may be applied to solving equations.

Consider, for instance, an algebraic equation of an odd degree (with real coefficients)

$$f(x) \equiv a_0 x^{2n+1} + a_1 x^{2n} + \ldots + a_{2n} x + a_{2n+1} = 0.$$

For sufficiently large absolute values of x the polynomial has the sign of the highest term, i.e. for positive x the sign of a_0, while for negative x the opposite sign. Since the polynomial is a continuous function, when it changes the sign it necessarily vanishes at an intermediate point. Hence we have the statement: every algebraic equation of an odd degree (with real coefficients) has at least one real root.

The Bolzano–Cauchy theorem can be used not only to establish the existence of a root but also to approximately calculate it. (This was the method used by Cauchy in proving the theorem which he gave in the chapter entitled "On numerical solution of equations"). We illustrate the above statement by an example. Let $f(x) = x^4 - x - 1$. Since $f(1) = -1$, $f(2) = 13$, the polynomial has a root between 1 and 2. Divide this interval [1, 2] into 10 equal parts by the points 1.1, 1.2, 1.3, ... and calculate

$$f(1.1) = -0.63 \ldots; \quad f(1.2) = -0.12 \ldots; \quad f(1.3) = +0.55 \ldots$$

We observe that the root lies between 1.2 and 1.3. Dividing also this interval into 10 parts we obtain

$$f(1.21) = -0.06\ldots;\ f(1.22) = -0.04\ldots;\ f(1.23) = +0.058\ldots;\ \ldots.$$

It is now clear that the root lies between 1.22 and 1.23; thus we already know the value of the root with the accuracy 0.01, etc.†.

70. Mean value theorem. The theorem proved in Sec. 69 can directly be generalized in the following way.

SECOND BOLZANO–CAUCHY THEOREM. *Let the function $f(x)$ be defined and continuous in the closed interval $[a, b]$; suppose that at the ends of this interval it has distinct values*

$$f(a) = A \quad \text{and} \quad f(b) = B.$$

Then, regardless of the value of the number C lying between A and B, a point c can be found between a and b such that

$$f(c) = C‡.$$

Proof. We take, for instance,

$$A < B, \quad \text{hence} \quad A < C < B.$$

Consider in the interval $[a, b]$ an auxiliary function $\varphi(x) = f(x) - C$. It is continuous in the interval and at its ends has opposite signs

$$\varphi(a) = f(a) - C = A - C < 0, \quad \varphi(b) = f(b) - C = B - C > 0.$$

Thus, by the first theorem, between a and b a point c can be found at which $\varphi(c) = 0$, i.e.

$$f(c) - C = 0 \quad \text{or} \quad f(c) = C.$$

This completes the proof.

Thus we have established an important property of the function $f(x)$ continuous in the interval: passing from one of its values to another one, the function takes every intermediate value at least once.

At first sight this property seems to imply the very essence of the continuity of a function. However, it is easy to construct discontinu-

† In fact, this way is, in practice, inconvenient, in view of the great amount of calculations involved; there exist methods which give the required result much faster (they are given in differential calculus).

‡ It is evident that the first Bolzano–Cauchy theorem is a particular case of the present one: viz. A and B have different signs, and $C = 0$.

ous functions which possess this property. For instance, the function [Sec. 67, *(3)*]

$$f(x) = \sin\frac{1}{x} \qquad (x \neq 0), \qquad f(0) = 0$$

in every interval containing the point $x = 0$ takes all possible values from -1 to $+1$†.

The above proved property of continuous functions implies the following (in fact, equivalent).

COROLLARY. *If a function $f(x)$ is defined and continuous in an interval $\mathcal{X}$ (closed or otherwise, finite or infinite), then the values taken by it also continuously fill an interval.*

Denote the set of values of the function $\{f(x)\}$ by $\mathcal{Y}$. Let

$$m = \inf \mathcal{Y}, \qquad M = \sup \mathcal{Y}‡$$

where l is an arbitrary number between m and M;

$$m < l < M.$$

We can always find values of the function, $f(x_1)$ and $f(x_2)$ (x_1 and x_2 belong to the interval $\mathcal{X}$), such that

$$m \leqslant f(x_1) < l < f(x_2) \leqslant M;$$

this follows from the very definition of the bounds of a numerical set. Then according to the theorem there exists between x_1 and x_2 a value $x = x_0$ (obviously also belonging to $\mathcal{X}$) such that $f(x_0)$ is equal exactly to l; consequently this number belongs to the set $\mathcal{Y}$.

Thus $\mathcal{Y}$ represents an interval with the ends m and M (which may or may not belong to the interval; cf. Sec. 73).

We know from Sec. 61 that in the case of a monotonic function the property of a function just formulated implies its continuity. The above example proves that this is not the case for all functions.

Remark. For the particular case when the considered function is a polynomial both theorems were announced much earlier than the general proof. For instance, for this case, Euler in his *Introduction to the Analysis of Infinitesi-*

† Not without reason Bolzano emphasized that this property is implied by continuity, but it cannot be taken as the basis of a definition of continuity.

‡ We remind the reader that if set $\mathcal{Y}$ is not bounded above (below), we agreed in Sec. 6 to assume $\sup \mathcal{Y} = +\infty$ ($\inf \mathcal{Y} = -\infty$).

mals presented a complete statement of the theorem of this subsection but without a convincing justification; the theorem then was applied to the solution of the problem of existence of real roots of algebraic equations [see Sec. 69]†. Euler, like other authors, sometimes employed geometric reasoning. Let us finally mention that Lagrange‡ began his *Treaty on Solving Numerical Equations of All Degrees* by an analytic proof (for a polynomial) of the theorem of Sec. 68, based on expansion into polynomials.

71. The existence of inverse functions. Let us apply the properties of a continuous function, deduced in the preceding subsection, to establishing some propositions on the existence of a single-valued inverse function and its continuity [cf. Sec. 23].

THEOREM. *Suppose that the function $y = f(x)$ is defined, increases (decreases) monotonically§ and is continuous in an interval $\mathcal{X}$. Then in the corresponding interval $\mathcal{Y}$ of the values of this function there exists a single-valued inverse function $x = g(y)$, also monotonically increasing (decreasing) and continuous.*

Proof. We confine ourselves to the case of an increasing function. We have already seen [see Corollary, Sec. 70] that the values of a continuous function $f(x)$ fill continuously an interval $\mathcal{Y}$, and hence for every value y_0 from this interval at least one value x_0 (from $\mathcal{X}$) can be found, such that

$$f(x_0) = y_0.$$

But, in view of the monotonicity of this function, there can be only one such value: if x is greater or smaller than x_0, then $f(x)$ is also greater or smaller than y_0, respectively.

Associating this value of x_0 with the arbitrary y_0 from $\mathcal{Y}$ we obtain a *single-valued* function

$$x = g(y),$$

inverse to the function $y = f(x)$.

It is readily observed that similarly to $f(x)$ this function $g(y)$ is also increasing monotonically. Let

$$y' < y'' \quad \text{and} \quad x' = g(y'), \quad x'' = g(y'');$$

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† pp. 44–46 of the Russian translation (see also footnote on p. 38).

‡ Joseph Louis Lagrange (1736–1813), an outstanding French mathematician.

§ In the strict sense of the word (this is essential here).

then, according to the definition of the function $g(y)$ we have, simultaneously,

$$y' = f(x') \quad \text{and} \quad y'' = f(x'').$$

If we had $x' > x''$, then by the increasing nature of the function $f(x)$ we would have also $y' > y''$, which contradicts the assumption. Neither can we have $x' = x''$ since then also $y' = y''$, which also contradicts the assumption. Thus, $x' < x''$ and $g(y)$ is increasing monotonically.

Finally, to prove the continuity of the function $x = g(y)$, we just use the theorem of Sec. 61, the conditions of which are satisfied: the considered function is monotonic and its values evidently fill the continuous interval $\mathcal{X}$†.

By means of this theorem we can again establish a number of familiar results.

For instance, applying it to the function x^n (where n is a positive integer) in the interval $\mathcal{X} = [0, +\infty)$ we deduce the existence and continuity of the (arithmetical) root

$$x = \sqrt[n]{y} \quad \text{for } y \text{ in } \quad \mathcal{Y} = [0, +\infty).$$

72. Theorem on the boundedness of a function. The fact that the function $f(x)$ is defined (so it takes finite values) for all values of x in a finite interval does not necessarily imply the boundedness of the function, i.e. the boundedness of the set $\{f(x)\}$ of the values it takes. For instance, the function

$$f(x) = \frac{1}{x}, \quad \text{if} \quad 0 < x \leqslant 1, \text{ and } f(0) = 0$$

takes only finite values but it is not bounded, since when x approaches zero the function can take arbitrarily large values. Incidentally, observe that in the semi-open interval (0,1] it is continuous but that at the point $x = 0$ it has a discontinuity.

The case is different for functions continuous in a closed interval.

FIRST THEOREM OF WEIERSTRASS. *If the function $f(x)$ is defined and continuous in a closed interval $[a, b]$, then it is bounded below*

† No matter what x from $\mathcal{X}$ we take, it is sufficient to set $y = f(x)$ in order that for this y the function $g(y)$ has as its value x.

and above, i.e. there exist constant and finite numbers m and M, such that

$$m \leqslant f(x) \leqslant M \quad \text{for} \quad a \leqslant x \leqslant b.$$

We begin the proof by assuming the converse: suppose that for x varying over the interval $[a, b]$ the function $f(x)$ is unbounded, say above.

Then for every positive integer n in the interval $[a, b]$ a value $x = x_n$ can be found, such that

$$f(x_n) \geqslant n. \tag{3}$$

By the Bolzano–Weierstrass lemma [Sec. 51], from the sequence $\{x_n\}$ a partial sequence $\{x_{n_k}\}$ can be extracted, which converges to a finite limit

$$x_{n_k} \to x_0 \quad (\text{as} \quad k \to +\infty),$$

and, obviously, $a \leqslant x_0 \leqslant b$. In view of the continuity of the function at point x_0 we have

$$f(x_{n_k}) \to f(x_0),$$

but this is impossible, since (3) implies that

$$f(x_{n_k}) \to +\infty.$$

This contradiction, together with a similar argument for the lower bound, proves the theorem.

73. The greatest and smallest values of a function. We know that an infinite numerical set, even bounded, may contain no greatest (smallest) element; if a function $f(x)$ is defined and even bounded in an interval of variation of x, then it may happen that there is no greatest (smallest) value in the set of values of the function $\{f(x)\}$. Then the least upper (greatest lower) bound of the values of the function $f(x)$ is not reached in the considered interval. This is, for instance, the case for the function

$$f(x) = x - E(x)$$

(its graph is given in Fig. 29). When x varies in any interval $[0, b]$ $(b \geqslant 1)$ the exact upper bound of values of the function is unity, but it is not reached; thus, the function has no greatest value.

The reader probably observes the connection between this fact and the presence of discontinuities of the considered function for

positive integral values of x. In fact, for functions continuous in a closed interval we have the

SECOND THEOREM OF WEIERSTRASS. *If a function $f(x)$ is defined and is continuous in a closed interval $[a, b]$, it attains in this interval its least upper and its greatest lower bounds.*

In other words, in the interval $[a, b]$ points x_0 and x_1 can be found, such that the values $f(x_0)$ and $f(x_1)$ are, respectively, the greatest and the smallest of all values of the function $f(x)$.

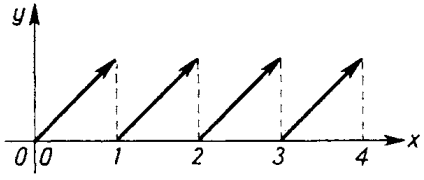

FIG. 29.

Proof. Set

$$M = \sup \{f(x)\};$$

according to the preceding theorem this is a finite number. Assume (with a view to obtaining a contradiction) that $f(x) < M$ for all x in $[a, b]$, i.e. the bound M is not attained. Then we can introduce an auxiliary function

$$\varphi(x) = \frac{1}{M - f(x)}.$$

Since, by the assumption, the denominator does not vanish, the function is continuous and consequently (in view of the preceding theorem) it is bounded, i.e. $f(x) \leqslant \mu$ $(\mu > 0)$. This readily implies that

$$f(x) \leqslant M - \frac{1}{\mu}.$$

In other words, the number $M - (1/\mu)$ smaller than M turns out to be the upper bound for the values of the function $f(x)$; this is impossible, since M is the least upper bound of these values. The contradiction so obtained proves the theorem: in the interval $[a, b]$ a value x_0 can be found, such that $f(x_0) = M$ is the greatest of all values of $f(x)$.

Similarly we can prove the result concerning the smallest value.

Observe that the above proof is "an existence proof". No means for computing, say, the value of $x = x_0$ have been given. Subse-

quently (Chapter 7, § 1), under more restrictive assumptions concerning the function, we shall learn how to actually calculate the values of the independent variable for which the function takes its greatest or smallest value.

If the function $f(x)$, when x varies over some interval $\mathfrak{X}$, is bounded, then the difference

$$\omega = M - m$$

is called its *oscillation* between the least upper and greatest lower bounds. Another definition of the oscillation is as follows: it is the least upper bound of the absolute values of the differences $f(x'') - f(x')$ where x' and x'' take independently arbitrary values in the interval $\mathfrak{X}$,

$$\omega = \sup_{x', x''} \{|f(x'') - f(x')|\}.$$

When we speak of a continuous function $f(x)$ in a closed finite interval $\mathfrak{X} = [a, b]$, then it follows from the theorem that the oscillation is simply the difference between the greatest and smallest values of the function in the considered interval.

In this case, the interval $\mathfrak{Y}$ of the values of the function is the closed interval $[m, M]$ and the oscillation is its length.

74. The concept of uniform continuity. If the function $f(x)$ is defined in an interval $\mathfrak{X}$ (closed or open, finite or infinite) and it is continuous at a point x_0 of this interval, then

$$\lim_{x \to x_0} f(x) = f(x_0)$$

or ("in the ε–δ language", Sec. 60): for any number $\varepsilon > 0$ a number $\delta > 0$ can be found, such that

$$|x - x_0| < \delta \quad \text{implies} \quad |f(x) - f(x_0)| < \varepsilon.$$

Assume now that the function $f(x)$ is continuous in the whole interval $\mathfrak{X}$, i.e. it is continuous at every point x_0 of this interval. Then, for every point x_0 of $\mathfrak{X}$ separately, for given ε a number δ can be found which corresponds to ε in the above sense. When x_0 varies within $\mathfrak{X}$, even if ε is fixed, the number δ, in general, also varies. It is obvious from Fig. 30 that the number δ applicable on the segment on which the function varies slowly (the graph is a shallow curve) can turn out to be too great for a segment of a fast variation of the

function (where the graph is steep). In other words, the number δ, in general, depends not only on ε but also on x_0.

If the number of values of x_0 were finite (for a fixed ε), then from the finite number of the corresponding numbers δ we could select the smallest and this would be valid for all considered points x_0 simultaneously.

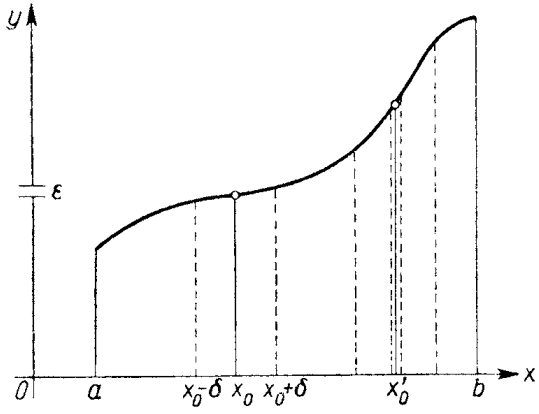

FIG. 30.

However, in the case of an infinite set of values of x_0 contained in the interval $\mathcal{X}$, the reasoning does not hold: they are associated (for a fixed ε) with an infinite set of numbers δ among which there can be also arbitrarily small ones. Thus, for a function $f(x)$ continuous in the interval $\mathcal{X}$, the following question arises: does there exist for a fixed ε a number δ which would be valid for all points x_0 from this interval?

If for any number $\varepsilon > 0$ a number $\delta > 0$ can be found, such that

$$|x - x_0| < \delta$$

implies

$$|f(x) - f(x_0)| < \varepsilon,$$

for arbitrary points x_0 and x within the considered interval $\mathcal{X}$, then the function $f(x)$ is said to be *uniformly continuous* in the interval $\mathcal{X}$.

In this case the number δ depends only on ε and may be prescribed before the choice of the point x_0, i.e. δ is valid for all x_0 simultaneously.

Uniform continuity means that in all parts of the interval one universal degree of nearness of two values of the argument is suf-

ficient to obtain a prescribed degree of nearness of the corresponding values of the function.

It may be shown by an example that continuity of a function at all points of an interval does not necessarily imply its uniform continuity in this interval. Let, for instance, $f(x) = \sin 1/x$ for x between 0 and $2/\pi$, excluding 0. In this case, the domain of variation of x is a non-closed interval $(0, 2/\pi]$ and at every point of it the function is continuous. Set now $x_0 = 2/(2n+1)\pi$, $x = 1/n\pi$ where n is any positive integer; then

$$f(x_0) = \sin(2n+1)\frac{\pi}{2} = \pm 1, \quad f(x) = \sin n\pi = 0.$$

Hence

$$|f(x) - f(x_0)| = 1,$$

although $|x - x_0| = 1/n(2n+1)\pi$ can be made arbitrarily small when n increases. Here for $\varepsilon = 1$ no δ can be found, which would be useful for all points x_0 in $(0, 2/\pi]$ simultaneously, although for every separate value of x, in view of the continuity of the function, such δ does exist.

75. Theorem on uniform continuity. It is very remarkable that in a closed interval $[a, b]$ such a result cannot occur; this follows from the following theorem.

CANTOR'S THEOREM†. *If a function $f(x)$ is defined and is continuous in a closed interval $[a, b]$, then it is also uniformly continuous in this interval.*

Proof. We assume the converse. Suppose that for a definite number $\varepsilon > 0$ such a number $\delta > 0$ does not exist (by this we mean the number considered in the definition of the uniform continuity). Then for any number $\delta > 0$ two values x and x' can be found in the interval $[a, b]$, such that

$$|x - x'| < \delta, \quad \text{but} \quad |f(x) - f(x')| \geqslant \varepsilon.$$

Take now the set $\{\delta_n\}$ of positive numbers, such that $\delta_n \to 0$. Then for every δ_n two values x_n and x_n' can be found in $[a, b]$ (they play the role of x and x') such that (for $n = 1, 2, 3, \ldots$)

$$|x_n - x_n'| < \delta_n, \quad \text{but} \quad |f(x_n) - f(x_n')| \geqslant \varepsilon.$$

† Georg Cantor (1845–1918)—a celebrated German mathematician, the originator of the modern theory of sets.

By virtue of the Bolzano–Weierstrass lemma [Sec. 51], from the bounded sequence $\{x_n\}$ a partial sequence can be extracted, which converges to a point x_0 of the interval $[a, b]$. In order not to complicate the notation we assume that the sequence $\{x_n\}$ itself already converges to x_0.

Since $x_n - x_n' \to 0$ (for $|x_n - x_n'| < \delta_n$ and $\delta_n \to 0$), the sequence $\{x_n'\}$ converges to x_0. Thus, in view of the continuity of the function at the point x_0 we should have

$$f(x_n) \to f(x_0) \quad \text{and} \quad f(x_n') \to f(x_0),$$

whence

$$f(x_n) - f(x_n') \to 0,$$

but this contradicts the fact that for all values of n

$$|f(x_n) - f(x_n')| \geqslant \varepsilon.$$

This completes the proof of the theorem.

This theorem directly implies the following corollary, which we shall require later.

COROLLARY. *Suppose that a function $f(x)$ is defined and continuous over the closed interval $[a, b]$. Then, corresponding to a given $\varepsilon > 0$, a number $\delta > 0$ can be found, such that if this interval be arbitrarily divided into subintervals, each with length smaller than δ, then in every such subinterval the oscillation of the function $f(x)$ is smaller than ε.*

In fact, if for the given $\varepsilon > 0$ we take for δ the number considered in the definition of the uniform continuity, then in the partial interval with length smaller than δ the absolute value of the difference between two arbitrary values of the function is smaller than ε. In particular, this is true also for the greatest and the smallest values, the difference of which gives the oscillation of the function in the considered interval [Sec. 73].

Thus, over a period of half a century, the basic properties of continuous functions were successively proved, beginning from the "obvious" ones and ending with the refined property of uniform continuity established in the last theorem. We emphasize once more that these proofs acquired the necessary strictness from the basis of the arithmetical theory of real numbers developed in the second half of the nineteenth century.

CHAPTER 5

DIFFERENTIATION OF FUNCTIONS OF ONE VARIABLE

§ 1. Derivative of a function and its computation

76. Problem of calculating the velocity of a moving point. Before proceeding to treat the foundations of the differential and integral calculus we draw the reader's attention to the fact that the ideas of calculus were originated as early as the seventeenth century, i.e. much earlier than the theories investigated in the preceding chapters. In the last chapter of this volume we shall survey the more important facts of the history of mathematical analysis and describe the merits of the two great mathematicians Newton and Leibniz, who completed the works of their predecessors by creating a really new calculus. In our discussion here we shall follow the modern demands of rigour, and not the history of the problem.

As an introduction to the differential calculus we shall examine in this subsection the problem of velocity, and in the next subsection the problem of finding a tangent to a curve; both problems are historically connected with the formation of the basic concept of the differential calculus, which was later called the *derivative*.

We begin by a simple example, namely we consider the free fall (in vacuum, when we can disregard the resistance of the air) of a heavy particle.

If the time t (seconds) is measured from the beginning of the fall, the distance covered s (metres) is given by the well-known formula

$$s = \frac{gt^2}{2}, \tag{1}$$

where $g = 9.81$ m/sec^2. From these facts it is required to determine the velocity v of motion of the point at a given instant of time t, when the point is located at M (Fig. 31).

Introduce an increment Δt of the variable t and consider the instant $t+\Delta t$ when the point is located at M_1. The increment MM_1 of the distance covered in the interval of time Δt we denote by Δs.

Substituting into (1) $t+\Delta t$ instead of t we obtain for the new value of distance the expression

$$s+\Delta s = \frac{g}{2}(t+\Delta t)^2,$$

whence

$$\Delta s = \frac{g}{2}(2t\cdot\Delta t+\Delta t^2).$$

Dividing Δs by Δt we obtain the *mean velocity* of fall of the point on the segment MM_1:

$$v_m = \frac{\Delta s}{\Delta t} = gt+\frac{g}{2}\Delta t.$$

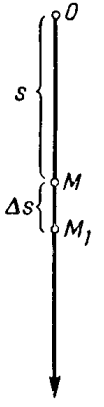

FIG. 31.

We observe that this velocity varies when Δt varies and the smaller the interval of time Δt elapsed from this instant, the better v_m describes the state of the falling point at the instant t.

By the *velocity* v of the point at the instant of time t we understand the limit to which the mean velocity v_m over the interval Δt tends, when Δt tends to zero.

Obviously, in our case

$$v = \lim_{\Delta t\to 0}\left(gt+\frac{g}{2}\Delta t\right) = gt.$$

In the same way we calculate the velocity v in the general case of, say, the rectilinear motion of a point. The location of the point is determined by its distance s measured from an initial point O. The time t is measured from an initial instant and it is not necessary that at this instant the point be that at which the point is located at O. The motion is regarded as completely determined when the equation of motion, $s = f(t)$, is known, by means of which we can find the position of the point at an arbitrary instant of time; in the considered example such a role is played by equation (1).

To determine the velocity v at a given instant of time t we would have as before to introduce an increment Δt of t; this is associated with the increase of the distance s by Δs. The ratio

$$\frac{\Delta s}{\Delta t}$$

yields the *mean velocity* v_m over the interval Δt. The instantaneous velocity v at the instant t is derived by passing to the limit

$$v = \lim_{\Delta t \to 0} v_m = \lim \frac{\Delta s}{\Delta t}.$$

We shall find later that another important problem leads to a similar limit operation.

77. Problem of constructing a tangent to a curve. Consider a curve (K) (Fig. 32) and a point M on it; let us establish the concept of a tangent to a curve at its point M.

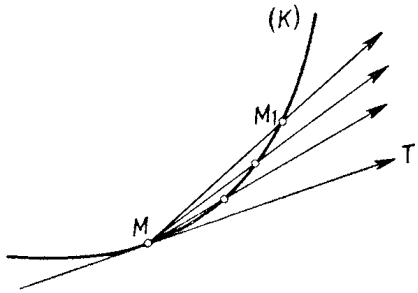

FIG. 32.

In the elementary course the tangent to a circle is defined as "the straight line cutting the curve in *one* common point". This definition, however, is of a particular nature and does not reveal

the essence of the problem. For instance, if we try to apply it to the parabola $y = ax^2$ (Fig. 33*a*), then at the origin of the coordinates O both coordinate axes satisfy the definition; but, as is probably clear to the reader, in fact only the x-axis is the tangent to the parabola at the point O.

We now proceed to give a general definition of the tangent. Take on the curve (K) (Fig. 32), besides point M, another point M_1 and construct the chord MM_1. When the point M_1 is displaced along the curve, this chord rotates about the point M.

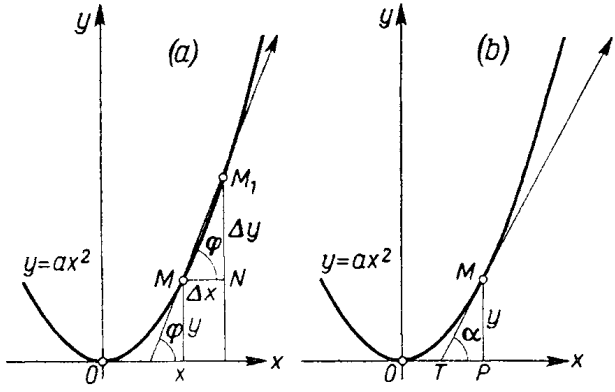

FIG. 33.

By the *tangent* to the curve (K) at a point M we understand the limiting position MT of the chord MM_1 when the point M_1 tends along the curve to coincide with the point M. The essence of the definition lies in the fact that the angle M_1MT tends to zero provided the chord MM_1 tends to zero.

Let us, for instance, apply this definition to the parabola $y = ax^2$ at an arbitrary point $M(x, y)$. Since the tangent passes through this point, to establish its position it is sufficient to know its slope. Our task therefore is to determine the slope $\tan\alpha$ of the tangent at point M.

Introducing an increment Δx of the abscissa x we pass from point M of the curve to a point M_1 with abscissa $x + \Delta x$ and ordinate

$$y + \Delta y = a(x + \Delta x)^2$$

(Fig. 33*a*). The slope $\tan\varphi$ of the chord MM_1 is determined from the right-angled triangle MNM_1. The side MN is equal to the increment of the abscissa Δx and evidently the side NM_1 is the corresponding increment of the ordinate

$$\Delta y = a(2x\Delta x + \Delta x^2),$$

whence

$$\tan\varphi = \frac{\Delta y}{\Delta x} = 2ax + a\Delta x.$$

To derive the slope of the tangent it is only necessary to pass to the limit $\Delta x \to 0$, since this is equivalent to the fact that the chord $MM_1 \to 0$. Then also $\varphi \to \alpha$ and (in view of the continuity of the function $\tan\varphi$) $\tan\varphi \to \tan\alpha$.

Thus, we have arrived at the following result:

$$\tan\alpha = \lim_{\Delta x \to 0} (2ax + a\Delta x) = 2ax^{\dagger}.$$

In the case of a curve with the equation

$$y = f(x)$$

the slope of the tangent is determined in a similar way. To the increment of the abscissa Δx there corresponds an increment Δy of the ordinate and the ratio

$$\frac{\Delta y}{\Delta x}$$

yields the slope of the chord, $\tan\varphi$. The slope of the tangent is now derived by passing to the limit,

$$\tan\alpha = \lim_{\Delta x \to 0} \tan\varphi = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

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† Incidentally we should observe that this implies a convenient way of actually constructing the tangent to a parabola. Namely, from $\triangle MPT$ (Fig. 33*b*) the segment

$$TP = \frac{y}{\tan\alpha} = \frac{ax^2}{2ax} = \frac{x}{2},$$

hence T is the centre of segment OP. Thus, to obtain the tangent to a parabola at its point M it suffices to divide into halves the segment OP and to connect its centre with point M.

78. Definition of the derivative. Comparing the operations carried out in solving the above fundamental problems, it is readily observed that in both cases, if we disregard the interpretation of the variables, in essence the same operation was performed: the increment of the function was divided by the increment of the independent variable and then the limit of their ratio was calculated. In this way we arrive at the basic concept of the analysis—the concept of *derivative.*

Suppose that the function $y = f(x)$ is defined in an interval $\mathcal{X}$. Consider a value $x = x_0$ of the independent variable and introduce an increment $\Delta x \gtrless 0$ remaining within the interval $\mathcal{X}$; thus the new value $x_0 + \Delta x$ also belongs to $\mathcal{X}$. Then the value $y_0 = f(x_0)$ of the function is replaced by a new value $y_0 + \Delta y = f(x_0 + \Delta x)$, i.e. we obtain the increment

$$\Delta y = \Delta f(x_0) = f(x_0 + \Delta x) - f(x_0).$$

The limit of the ratio of the increment of the function Δy to the increment of the independent variable Δx which produced the former, when Δx tends to zero, i.e.

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x},$$

is called the *derivative*† of the function $y = f(x)$ with respect to the independent variable x for its given value (or at a given point) $x = x_0$.

Thus, the derivative for a given value $x = x_0$, if it exists, is a definite number‡. If now the derivative exists in the whole interval $\mathcal{X}$, i.e. for every value of x in this interval, then it is a function of x.

Making use of this concept, we can state the fact of Sec. 76 about the velocity of a moving point as follows:

The velocity v is the derivative of the distance travelled with respect to the time t.

If the word "velocity" be understood in a more general sense, we could always regard the derivative as a certain "velocity". Namely, given a function y of the independent variable x we may formulate

† The term "derivative" was introduced by Lagrange at the turn of the eighteenth century.

‡ We confine ourselves for the time being to the case when the limit is finite [see Sec. 87].

the problem of the *velocity*† *of change of the variable* y *as compared with the variable* x (for a given value of the latter).

If the increment Δx of x produces an increment Δy of y, then as in Sec. 76 by *mean velocity* of the change of y compared with x, when x changes by the quantity Δx, we may regard the ratio

$$V_{\mathrm{m}} = \frac{\Delta y}{\Delta x}.$$

It is natural to call the velocity of the change of y for a given value of x the limit of this ratio when Δx tends to zero

$$V = \lim_{\Delta x \to 0} V_{\mathrm{m}} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x},$$

i.e. the derivative of y with respect to x.

In Sec. 77 we considered a curve given by the equation $y = f(x)$ and we solved the problem of constructing the tangent to it at a given point. Now we can formulate the derived result as follows:

The slope $\tan\alpha$ of the tangent is the derivative of the ordinate y with respect to the abscissa x.

This geometric interpretation of the derivative is frequently useful.

Further, let us give a few examples illustrating the concept of derivative.

If the velocity of motion v is not constant and varies in the course of time, i.e. $v = f(t)$, then we may consider the "velocity of change of the velocity", calling it the *acceleration.*

Namely, if to the increment of time Δt there corresponds the increment Δv of the velocity, then the ratio

$$a_{\mathrm{m}} = \frac{\Delta v}{\Delta t}$$

gives the mean acceleration over the interval of time Δt and its limit yields the acceleration of the motion at the considered instant of time

$$a = \lim_{\Delta t \to 0} a_{\mathrm{m}} = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}.$$

† The word *rate* is often used instead of velocity. [Ed.]

Thus, the *acceleration is the derivative of the velocity with respect to time.*

Consider now a "linear" continuous distribution of mass along a rectilinear segment (i.e. actually along a rod the width and thickness of which is neglected). Let the location of a point on this segment be determined by the abscissa x measured (for instance in cm) from the beginning of the segment. The mass m distributed over the segment $[0, x]$ depends on x, i.e. $m = f(x)$. The increment Δx of the abscissa of the end of the segment results in an increment Δm of the mass; in other words, Δm is the mass of the segment $[x, x+\Delta x]$, adjacent to the point x. Then the mean density of the distribution of mass over the considered segment is given by the ratio

$$\varrho_{\mathrm{m}} = \frac{\Delta m}{\Delta x}$$

The limit of this mean density when the segment contracts to a point, i.e. when $\Delta x \to 0$,

$$\varrho = \lim_{\Delta x \to 0} \varrho_{\mathrm{m}} = \lim_{\Delta x \to 0} \frac{\Delta m}{\Delta x}.$$

is called the (linear) *density at the point* x: this density is the derivative of the mass with respect to the abscissa.

We consider now the theory of heat; by means of the derivative we shall introduce the concept of heat capacity at a given temperature.

Denote the relevant physical quantities in the following way: θ is the temperature (in °C), W the amount of heat which should be supplied to the body in heating it from 0° to θ° (in calories). It is clear that W is a function of θ, $W = f(\theta)$. Let us introduce an increment $\Delta\theta$ of θ; then W also acquires an increment ΔW. The mean *heat capacity* in heating from θ° to $(\theta + \Delta\theta)$° is

$$c_{\mathrm{m}} = \frac{\Delta W}{\Delta \theta}.$$

But since, in general, this mean heat capacity varies with $\Delta\theta$ we cannot regard it as the *heat capacity at a given temperature* θ. To derive the latter we pass to the limit

$$c = \lim_{\Delta\theta \to 0} c_{\mathrm{m}} = \lim \frac{\Delta W}{\Delta \theta}.$$

Thus, we can state that the heat capacity of a body is the derivative of the amount of heat with respect to temperature.

All applications of the derivative (their number can easily be increased) clearly indicate that the derivative is essentially connected with the fundamental concepts of various branches of science, often assisting in the establishing of these concepts.

The calculation of derivatives and the investigation of their properties constitute the basic contents of the differential calculus.

For denoting the derivative various symbols are being employed, namely

$$\frac{dy}{dx} \quad \text{or} \quad \frac{df(x_0)}{dx}^{\dagger} \qquad \text{(Leibniz)},$$

$$y' \quad \text{or} \quad f'(x_0) \qquad \text{(Lagrange)},$$

$$Dy \quad \text{or} \quad Df(x_0) \qquad \text{(Cauchy)}.$$

We shall mostly use the simple notation of Lagrange. If the functional notation is employed (of the second column) the letter x_0 in parenthesis indicates the value of the independent variable for which the derivative is calculated. We observe finally that in the cases when doubt can arise concerning the variable with respect to which the derivative is taken (in comparison with which the "velocity of change of the function" is determined), this variable is indicated by a subscript

$$y'_x, \quad f'_x(x_0), \quad D_x y, \quad D_x f(x_0),$$

the subscript x being not connected with the particular value x_0 of the independent variable for which the derivative is calculated.

(In a sense we may say that the whole symbols

$$\frac{df}{dx}, f' \quad \text{or} \quad f'_x, Df \quad \text{or} \quad D_x f$$

play the role of a *functional notation* for the derivative of a function.)

† For the time being we regard Leibniz's notations as whole symbols; later we shall find that they may be regarded also as fractions. We shall not employ Newton's notation $\dot{y}$ which assumes that the independent variable is time (cf. Sec. 224).

We now write, making use of these symbols, some of the derived results. For the velocity of motion we have

$$v = \frac{ds}{dt} \quad \text{or} \quad v = s'_t,$$

and for the acceleration

$$a = \frac{dv}{dt} \quad \text{or} \quad a = v'_t.$$

Similarly, the slope of the tangent to the curve $y = f(x)$ has the form

$$\tan\alpha = \frac{dy}{dx} \quad \text{or} \quad \tan\alpha = y'_x.$$

Similarly in other cases.

79. Examples of the calculation of the derivative. We consider the derivatives of some elementary functions.

(1) First observe that the following obvious results are true: if $y = c = \text{const}$, then $\Delta y = 0$ for an arbitrary Δx and hence $y' = 0$; if $y = x$ then $\Delta y = \Delta x$ and $y' = 1$.

(2) Power function. $y = x^\mu$ (where μ is an arbitrary real number). The domain of variation of x depends on μ; this was shown in Sec. 22, *(2)*. We have (for $x \neq 0$)

$$\frac{\Delta y}{\Delta x} = \frac{(x+\Delta x)^\mu - x^\mu}{\Delta x} = x^{\mu-1} \cdot \frac{\left(1+\dfrac{\Delta x}{x}\right)^\mu - 1}{\dfrac{\Delta x}{x}}.$$

Making use of the limit computed in Sec. 65, *(3)* we obtain

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \mu x^{\mu-1}.\dagger$$

In particular,

$$\text{if} \quad y = \frac{1}{x} = x^{-1}, \quad \text{then} \quad y' = (-1)x^{-2} = -\frac{1}{x^2},$$

$$\text{if} \quad y = \sqrt{x} = x^{\frac{1}{2}}, \quad \text{then} \quad y' = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}.$$

† If $\mu > 0$, then for $x = 0$ it is easy to deduce directly the value of the derivative: $y' = 0$.

(3) Exponential function. $y = a^x$ $(a > 0, -\infty < x < +\infty)$. Here

$$\frac{\Delta y}{\Delta x} = \frac{a^{x+\Delta x} - a^x}{\Delta x} = a^x \frac{a^{\Delta x} - 1}{\Delta x}.$$

Making use of the limit of Sec. 65, *(2)* we obtain

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = a^x \log a.$$

In particular,

$$\text{if} \quad y = e^x, \quad \text{then} \quad y' = e^x.$$

Thus, the rate of increase of the exponential function (for $a > 1$) is proportional to the value of the function itself: the greater the value reached by the function, the faster it increases. This is a characteristic of the growth of the exponential function.

(4) Logarithmic function. $y = \log_a x$ $(0 < a \neq 1, 0 < x < +\infty)$. In this case

$$\frac{\Delta y}{\Delta x} = \frac{\log_a(x+\Delta x) - \log_a x}{\Delta x} = \frac{1}{x} \frac{\log_a\left(1 + \frac{\Delta x}{x}\right)}{\frac{\Delta x}{x}}.$$

Making use of the limit of Sec. 65, *(1)* we have

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{\log_a e}{x}.$$

In particular, for the natural logarithm we obtain the following particularly simple result:

$$\text{if} \quad y = \log x, \quad \text{then} \quad y' = \frac{1}{x}.$$

This is the basis (although actually not new) of the preference for natural logarithms in theoretical investigations.

The fact that the velocity of increase of the logarithmic function (for $a > 1$) is inversely proportional to the value of the argument, and being positive tends to zero when the argument increases to infinity, is a characteristic of the growth of the logarithmic function.

(5) Trigonometric functions. Let $y = \sin x$; then

$$\frac{\Delta y}{\Delta x} = \frac{\sin(x+\Delta x) - \sin x}{\Delta x} = \frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} \cos\left(x + \frac{\Delta x}{2}\right).$$

Making use of the continuity of the function $\cos x$ and the familiar [Sec. 34, *(5)*] limit $\lim\limits_{\alpha\to 0} \sin\alpha/\alpha = 1$ we obtain

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \cos x^{\dagger}.$$

Similarly, we find

$$\text{if } y = \cos x, \text{ then } y' = -\sin x.$$

For $y = \tan x$ we have

$$\frac{\Delta y}{\Delta x} = \frac{\tan(x+\Delta x) - \tan x}{\Delta x} = \frac{\frac{\sin(x+\Delta x)}{\cos(x+\Delta x)} - \frac{\sin x}{\cos x}}{\Delta x}$$

$$= \frac{\sin(x+\Delta x)\cos x - \cos(x+\Delta x)\sin x}{\Delta x \cdot \cos x \cdot \cos(x+\Delta x)}$$

$$= \frac{\sin \Delta x}{\Delta x} \cdot \frac{1}{\cos x \cdot \cos(x+\Delta x)}.$$

Hence, as before

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{1}{\cos^2 x} = \sec^2 x$$

and also

$$\text{if } y = \cot x, \text{ then } y' = -\frac{1}{\sin^2 x} = -\operatorname{cosec}^2 x.$$

80. Derivative of the inverse function. Before proceeding to the calculation of the derivative of inverse trigonometric functions we prove the following theorem.

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† Observe that this formula owes its simplicity to the fact that the angle is measured in radians. If x were measured, say, in degrees, the limit of the ratio of the sine to the angle would be equal not to unity but, as readily observed to $\pi/180$, and we would have

$$(\sin x)' = \frac{\pi}{180} \cos x.$$

THEOREM. *Suppose that (1) the function $f(x)$ satisfies the conditions of the theorem of Sec. 71 on the existence of the inverse function; (2) it has at the point $x = x_0$ a finite and non-zero derivative $f'(x_0)$. Then the derivative of the inverse function $x = g(y)$ exists at the corresponding point $y_0 = f(x_0)$, and is equal to $1/f'(x_0)$.*

Proof. For an arbitrary increment Δy of $y = y_0$, the function $x = g(y)$ acquires a corresponding increment Δx. We observe that if $\Delta y \neq 0$ then, from the single-valuedness of the function $y = f(x)$, $\Delta x \neq 0$. We have

$$\frac{\Delta x}{\Delta y} = \frac{1}{\dfrac{\Delta y}{\Delta x}}.$$

If now $\Delta y \to 0$ according to any law, then in view of the continuity of the function $x = g(y)$, the increment $\Delta x \to 0$ as well. But then the denominator of the right-hand side of the above relation tends to the limit $f'(x_0) \neq 0$, and consequently, there exists a limit of the left-hand side equal to the inverse $1/f'(x_0)$; this is the derivative $g'(y_0)$.

Thus, we have the simple formula

$$x'_y = \frac{1}{y'_x}.$$

It is easy to find its geometric meaning. We know that the derivative y'_x is the tangent of the angle α made by the tangent to the graph of the function $y = f(x)$ with the x-axis. But the inverse function $x = g(y)$ has the same graph, only the independent variable is now on the y-axis. Hence the derivative x'_y is equal to the tangent of the angle β made by the same tangent with the y-axis (Fig. 34). Thus, the derived formula is reduced to the familiar relation

$$\tan\beta = 1/\tan\alpha,$$

connecting the tangents of two angles α and β the sum of which is $\pi/2$.

Let, for instance, $y = a^x$. The inverse function is $x = \log_a y$. Since (see *(3)*) $y'_x = a^x \cdot \log a$, by our formula

$$x'_y = \frac{1}{y'_x} = \frac{1}{a^x \cdot \log a} = \frac{\log_a e}{y},$$

in accordance with *(4)*.

We now proceed to calculate derivatives of inverse trigonometric functions; for convenience we exchange the variables x and y; then we write the derived formula in the form

$$y'_x = \frac{1}{x'_y}.$$

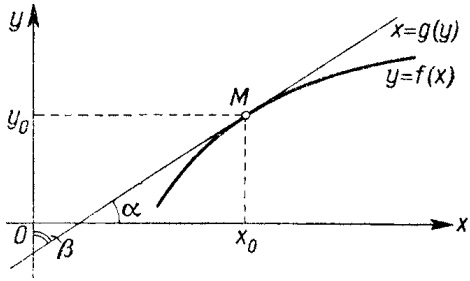

FIG. 34.

(6) Inverse trigonometric functions. Consider the function $y = \arcsin x$ $(-1 < x < 1)$, where $-\pi/2 < y < \pi/2$. It is the inverse of the function $x = \sin y$ which has for the considered values of y a positive derivative $x'_y = \cos y$. Then there also exists the derivative y'_x given in accordance with our formula,

$$y'_x = \frac{1}{x'_y} = \frac{1}{\cos y} = \frac{1}{\sqrt{(1-\sin^2 y)}} = \frac{1}{\sqrt{(1-x^2)}};$$

the root is taken with the positive sign, since $\cos y > 0$.

We exclude the values $x = \pm 1$, since for the corresponding values $y = \pm \pi/2$ the derivative $x'_y = \cos y = 0$.

The function $y = \operatorname{arc\,tan} x (-\infty < x < +\infty)$ is the inverse of the function $x = \tan y$. According to our formula,

$$y'_x = \frac{1}{x'_y} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}.$$

Similarly, we obtain

for $y = \arccos x$

$$y' = -\frac{1}{\sqrt{(1-x^2)}} \qquad (-1 < x < 1),$$

for $y = \operatorname{arc\,cot} x$

$$y' = -\frac{1}{1+x^2} \qquad (-\infty < x < +\infty).$$

81. Summary of formulae for derivatives. We collect together all the formulae we have so far derived:

1. $y = c$, $y' = 0$;
2. $y = x$, $y' = 1$;
3. $y = x^\mu$, $y' = \mu x^{\mu-1}$;

   $y = \frac{1}{x}$, $y' = -\frac{1}{x^2}$;

   $y = \sqrt{x}$, $y' = \frac{1}{2\sqrt{x}}$;
4. $y = a^x$, $y' = a^x \cdot \log a$;

   $y = e^x$, $y' = e^x$;
5. $y = \log_a x$, $y' = \frac{\log_a e}{x}$;

   $y = \log x$, $y' = \frac{1}{x}$;
6. $y = \sin x$, $y' = \cos x$;
7. $y = \cos x$, $y' = -\sin x$;
8. $y = \tan x$, $y' = \sec^2 x = \frac{1}{\cos^2 x}$;
9. $y = \cot x$, $y' = -\operatorname{cosec}^2 x = -\frac{1}{\sin^2 x}$;
10. $y = \operatorname{arc}\sin x$, $y' = \frac{1}{\sqrt{(1-x^2)}}$;
11. $y = \operatorname{arc}\cos x$, $y' = -\frac{1}{\sqrt{(1-x^2)}}$;
12. $y = \operatorname{arc}\tan x$, $y' = \frac{1}{1+x^2}$;
13. $y = \operatorname{arc}\tan x$, $y' = -\frac{1}{1+x^2}$.

82. Formula for the increment of a function. We shall prove here two simple propositions which we shall require later.

Suppose that the function $y = f(x)$ is defined in an interval $\mathfrak{X}$. For a definite value $x = x_0$ of this interval, denote by $\Delta x \gtrless 0$ an arbitrary increment of x subject only to the condition that the point

$x_0 + \Delta x$ remains within $\mathfrak{X}$. Then the corresponding increment of the function is

$$\Delta y = \Delta f(x_0) = f(x_0 + \Delta x) - f(x_0).$$

(1) If the function $y = f(x)$ at the point x_0 has a (finite) derivative $y'_x = f'(x_0)$, then the increment of the function can be represented in the form

$$\Delta f(x_0) = f'(x_0) \cdot \Delta x + \alpha \cdot \Delta x \tag{2}$$

or, more briefly

$$\Delta y = y'_x \cdot \Delta x + \alpha \cdot \Delta x, \tag{2a}$$

where α is a quantity depending on Δx and tending to zero when Δx tends to zero.

Since, by the definition of the derivative, for $\Delta x \to 0$,

$$\frac{\Delta y}{\Delta x} \to y'_x,$$

assuming

$$\alpha = \frac{\Delta y}{\Delta x} - y'_x,$$

we see that also $\alpha \to 0$. Determining Δy we arrive at formula (2a).

Since the quantity $\alpha \cdot \Delta x$ (for $\Delta x \to 0$) is an infinitesimal of a higher order than Δx, making use of the notation introduced in Sec. 54, we can rewrite our formulae in the form

$$\Delta f(x_0) = f'(x_0) \cdot \Delta x + o(\Delta x) \tag{3}$$

or

$$\Delta y = y'_x \cdot \Delta x + o(\Delta x). \tag{3a}$$

Remark. So far we have assumed that $\Delta x \gtrless 0$; the quantity α has not been defined for $\Delta x = 0$. When we said that $\alpha \to 0$ for $\Delta x \to 0$, then (as before) we assumed that Δx tends to zero according to some arbitrary law, but does not take the value zero. Now set $\alpha = 0$ when $\Delta x = 0$; evidently formula (2) is now valid also for $\Delta x = 0$. Thus, the relation

$$\alpha \to 0 \quad \text{as} \quad \Delta x \to 0$$

can be understood in a wider sense than before—without excluding the possibility of Δx tending to zero through values including zero.

The above formulae imply the following:

(2) If the function $y=f(x)$ at the point x_0 has a (finite) derivative, then at this point the function is necessarily continuous.

In fact, it is clear from (2a) that this relation for $\Delta x\to 0$ implies $\Delta y\to 0$.

83. Rules for the calculation of derivatives. In the preceding subsections we calculated the derivatives of the elementary functions. Here and in the next subsection we shall establish a number of simple rules, by means of which it is possible to calculate the derivative of an arbitrary function constructed from the elementary functions by means of a finite number of arithmetical operations and superpositions [Sec. 25].

I. Suppose that the function $u=\varphi(x)$ has (at a definite point x) the derivative u'. We prove that the function $y=cu$ $(c=\text{const})$ also has a derivative (at the same point) and we shall calculate it.

If the independent variable x acquires an increment Δx, then the function u acquires an increment Δu passing from the former value u to the new value $u+\Delta u$. The new value of the function y is $y+\Delta y=c(u+\Delta u)$. Hence $\Delta y=c.\Delta u$ and

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=c\cdot\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}=c\cdot u'.$$

Thus the derivative exists and is equal to

$$y'=(c\cdot u)'=c\cdot u'.$$

This formula expresses the following rule: *a constant factor can be taken outside the symbol of the derivative operator.*

II. Suppose that the functions $u=\varphi(x)$, $v=\psi(x)$ have (at a definite point) the derivatives u', v'. We prove that the function $y=u\pm v$ also has a derivative (at the same point) and we shall calculate it.

Introduce an increment Δx of x; then u, v and y acquire the increments Δu, Δv, Δy respectively. Their new values $u+\Delta u$, $v+\Delta v, y+\Delta y$ are connected by the same relation

$$y+\Delta y=(u+\Delta u)\pm(v+\Delta v).$$

Hence

$$\Delta y=\Delta u\pm\Delta v,\quad \frac{\Delta y}{\Delta x}=\frac{\Delta u}{\Delta x}\pm\frac{\Delta v}{\Delta x}$$

and

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \pm \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} = u' \pm v'.$$

Thus, the derivative y' exists and is equal to

$$y' = (u \pm v)' = u' \pm v'.$$

This result can easily be generalized to an arbitrary number of terms (exactly by the same method).

III. Under the same assumptions with respect to the functions u, v we prove that the function $y = u \cdot v$ also has a derivative and we shall find it.

As before, the increment Δx is associated with the increments $\Delta u, \Delta v$ and Δy; also $y + \Delta y = (u + \Delta u)(v + \Delta v)$ and hence

$$\Delta y = \Delta u \cdot v + u \cdot \Delta v + \Delta u \cdot \Delta v$$

and

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} \cdot v + u \cdot \frac{\Delta v}{\Delta x} + \frac{\Delta u}{\Delta x} \cdot \Delta v.$$

Since $\Delta x \to 0$ by Sec. 82 we have $\Delta v \to 0$, and so

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \cdot v + u \cdot \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} = u' \cdot v + u \cdot v',$$

i.e. the derivative y' exists,

$$y' = (u \cdot v)' = u' \cdot v + u \cdot v'.$$

If $y = uvw$ and u', v', w' exist, then

$$y' = [(uv) \cdot w]' = (uv)' \cdot w + (uv) \cdot w' = u'vw + uv'w + uvw'.$$

It is readily observed that for the case of n factors we have similarly

$$\overbrace{[uvw \ldots s]}^{n}{}' = u'vw \ldots s + uv'w \ldots s + uvw' \ldots s + \ldots + uvw \ldots s'.$$

To prove this result we may use the method of mathematical induction.

IV. Finally, if u, v satisfy the former assumptions and moreover v does not vanish, we prove that the function $y = u/v$ also has a derivative and we find it.

Employing the same notation as before we have

$$y+\Delta y=\frac{u+\Delta u}{v+\Delta v},$$

whence

$$\Delta y=\frac{\Delta u\cdot v-u\cdot\Delta v}{v(v+\Delta v)} \quad \text{and} \quad \frac{\Delta y}{\Delta x}=\frac{\frac{\Delta u}{\Delta x}\cdot v-u\cdot\frac{\Delta v}{\Delta x}}{v(v+\Delta v)}.$$

When Δx tends to zero (so at the same time $\Delta v\to 0$) we obtain the derivative

$$y'=\left(\frac{u}{v}\right)'=\frac{u'\cdot v-u\cdot v'}{v^2}.$$

84. Derivative of a compound function. We are now in a position to establish a very important rule which makes it possible for us to calculate in practical cases the derivative of a compound function if the derivatives of the functions of which it is constructed are known.

V. Suppose that: *(1)* the function $u=\varphi(x)$ has at a point x_0 the derivative $u'_x=\varphi'(x_0)$; *(2)* the function $y=f(u)$ has at the corresponding point $u_0=\varphi(x_0)$ the derivative $y'_u=f'(u)$. Then the compound function $y=f(\varphi(x))$ has at the considered point x_0 a derivative as well, which is equal to the product of the derivatives of the functions $f(u)$ and $\varphi(x)$:

$$[f(\varphi(x))]'=f'_u(\varphi(x_0))\cdot\varphi'(x_0)^{\dagger}$$

or more briefly

$$y'_x=y'_u\cdot u'_x.$$

To prove this we introduce an arbitrary increment Δx of x; let Δu be the corresponding increment of the function $u=\varphi(x)$ and finally let Δy be the increment of the function $y=f(u)$ corresponding to the increment Δu. Making use of relation (2a) and replacing x by u we have

$$\Delta y=y'_u\cdot\Delta u+\alpha\cdot\Delta u$$

† We emphasize that the symbol $f'_u(\varphi(x_0))$ denotes the derivative of the function $f(u)$ *with respect to its argument* u (*not with respect to* x), calculated for the value $u_0=\varphi(x_0)$ of this argument.

(α depends on Δu and tends to zero as the latter tends to zero). Dividing throughout by Δx we obtain

$$\frac{\Delta y}{\Delta x} = y'_u \cdot \frac{\Delta u}{\Delta x} + \alpha \cdot \frac{\Delta u}{\Delta x}.$$

If Δx tends to zero, Δu also tends to zero [Sec. 82, *(2)*], and we know that then the quantity α depending on Δu also tends to zero. Consequently, there exists the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = y'_u \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = y'_u \cdot u'_x,$$

which gives the required derivative y'_x.

Remark. Here the remark of Sec. 82 is useful (which concerns the quantity α when $\Delta x = 0$): as long as Δx were the increment of the independent variable we could assume Δx to be distinct from zero, but when Δx is replaced by the increment of the function $u = \varphi(x)$, then even for $\Delta x \neq 0$ we are not allowed to assume that $\Delta u \neq 0$.

85. Examples†. We now give a few examples of application of the rules I–IV.

(1) Consider the polynomial

$$y = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-2} x^2 + a_{n-1} x + a_n.$$

By rule II and then I we have

$$\begin{aligned} y' &= (a_0 x^n)' + (a_1 x^{n-1})' + \ldots + (a_{n-2} x^2)' + (a_{n-1} x)' + (a_n)' \\ &= a_0 (x^n)' + a_1 (x^{n-1})' + \ldots + a_{n-2} (x^2)' + a_{n-1} (x)' + (a_n)'. \end{aligned}$$

Making use of formulae 1, 2, 3 [Sec. 81] we finally obtain

$$y' = n a_0 x^{n-1} + (n-1) a_1 x^{n-2} + \ldots + 2 a_{n-2} x + a_{n-1}.$$

(2) $y = (2x^2 - 5x + 1) \cdot e^x$.

According to rule II

$$y' = (2x^2 - 5x + 1)' \cdot e^x + (2x^2 - 5x + 1) \cdot (e^x)'.$$

From the preceding example and formula 4 [Sec. 81] we find that

$$y' = (4x - 5) \cdot e^x + (2x^2 - 5x + 1) \cdot e^x = (2x^2 - x - 4) \cdot e^x.$$

(3) $y = \dfrac{x \sin x + \cos x}{x \cos x - \sin x}$.

---

† The letters x, y, u, v here denote the variables and the other letters constant quantities.

Here we first use rule IV and then II and III [and formulae 6, 7, Sec. 81].

$$y' = \frac{(x\sin x+\cos x)'(x\cos x-\sin x)-(x\sin x+\cos x)(x\cos x-\sin x)'}{(x\cos x-\sin x)^2}$$

$$= \frac{x\cos x(x\cos x-\sin x)-(x\sin x+\cos x)(-x\sin x)}{(x\cos x-\sin x)^2}$$

$$= \frac{x^2}{(x\cos x-\sin x)^2}.$$

The derivatives of the numerator and the denominator have been calculated without dividing it into elementary operations. By experience it is necessary to become able to write down derivatives directly.

The following are examples on the calculation of derivatives of compound functions.

(4) Let $y = \log\sin x$, i.e. $y = \log u$ where $u = \sin x$. By rule V, $y'_x = y'_u u'_x$. The derivative $y'_u = (\log u)'_u = 1/u$ (formula 5) should be taken for $u = \sin x$. Thus,

$$y'_x = \frac{1}{\sin x}(\sin x)' = \frac{\cos x}{\sin x} = \cot x \quad \text{(formula 6)}.$$

(5) $y = e^{x^2}$, i.e. $y = e^u$, where $u = x^2$;

$$y'_x = e^{x^2}\cdot(x^2)' = 2x\cdot e^{x^2} \qquad \text{(V; 4 and 3)}$$

Of course, there is no need to write down the component functions separately.

(6) $y = \sin ax$; $y'_x = \cos ax\cdot(ax)' = a\cdot\cos ax$ (V; 7, I, 2)

(7) $y = \arctan\frac{1}{x}$; $\quad y'_x = \frac{1}{1+\frac{1}{x^2}}\left(\frac{1}{x}\right)' = \frac{x^2}{1+x^2}\left(-\frac{1}{x^2}\right)$

$$= -\frac{1}{1+x^2} \qquad \text{(V; 12, 3)}$$

The case of a compound function obtained by several superpositions can be tackled by successive applications of rule V:

(8) $y = \sqrt{(\tan\frac{1}{2}x)}$;
then

$$y'_x = \frac{1}{2\sqrt{(\tan\frac{1}{2}x)}}(\tan\tfrac{1}{2}x)'_x \qquad \text{(V; 3)}$$

$$= \frac{1}{2\sqrt{(\tan\frac{1}{2}x)}}\sec^2\tfrac{1}{2}x\,(\tfrac{1}{2}x)'_x \qquad \text{(V; 8)}$$

$$= \frac{\sec^2\frac{1}{2}x}{4\sqrt{(\tan\frac{1}{2}x)}}.$$

Let us consider a few more examples of the application of these rules:

(9) $y = \log[x + \sqrt{(x^2 + c)}]$; $\quad y'_x = \dfrac{1}{x + \sqrt{(x^2+c)}}\,[x + \sqrt{(x^2 + c)}]'_x$

$$= \frac{1}{x + \sqrt{(x^2 + c)}}\left(1 + \frac{x}{\sqrt{(x^2 + c)}}\right) = \frac{1}{\sqrt{(x^2 + c)}}.$$

(10) $y = \dfrac{x}{c\sqrt{(x^2 + c)}}$; $\quad y' = \dfrac{1}{c}\,\dfrac{1 \cdot \sqrt{(x^2 + c)} - x\,\dfrac{x}{\sqrt{(x^2 + c)}}}{(\sqrt{(x^2 + c)})^2}$

$$= \frac{1}{(x^2 + c)^{3/2}}.$$

(11) As an exercise let us examine the problem of the derivative of a power-exponential expression $y = u^v$ $(u > 0)$ where u and v are functions of x, having at the considered point the derivatives u', v'.

Taking logarithms in the relation $y = u^v$ we obtain

$$\log y = v \log u. \tag{4}$$

Thus, the expression for y can be rewritten in the form $y = e^{v \log u}$ which implies that the derivative y' exists. The calculation itself can most simply be carried out by equating the derivatives with respect to x of both sides of relation (4). In doing so we make use of rules V and III (bearing in mind that u, v and y are functions of x). Thus we obtain

$$\frac{1}{y}\,y' = v' \log u + v\,\frac{1}{u}\,u',$$

whence

$$y' = y\left(\frac{vu'}{u} + v' \log u\right).$$

Replacing y by its expression,

$$y' = u^v\left(\frac{vu'}{u} + v' \log u\right). \tag{5}$$

This formula was first established by Leibniz and J. Bernoulli.

For instance

$$\text{if } y = x^{\sin x}, \quad \text{then} \quad y'_x = x^{\sin x}\left(\frac{\sin x}{x} + \cos x \cdot \log x\right).$$

86. One-sided derivatives. We finally examine the exceptional cases which may occur for derivatives. We begin by establishing the concept of *one-sided derivatives.* If the value of x to be considered is one of the end-points of the interval $\mathfrak{X}$ over which the function $y = f(x)$ is defined, then in calculating the limit of $\Delta y/\Delta x$ we have

to confine ourselves to Δx tending to zero only from the right (when we consider the left-hand end of the interval) or from the left (for the right-hand end). In this case we speak of the *one-sided* derivative, from the right or from the left. At the corresponding points on the graph, the function has a *one-sided tangent.*

It can also occur that for an interior point x there exist only one-sided limits of the ratio $\Delta y/\Delta x$ (for $\Delta x \to +0$ or $\Delta x \to -0$), which are not equal; they are also called *one-sided derivatives.* For the graph of the function there exist, at the considered point only, *one-sided* tangents inclined at an angle to each other: the point is a "sharp" point (Fig. 35).

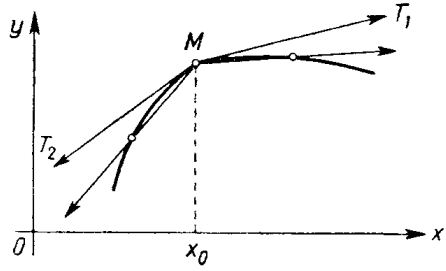

FIG. 35.

As an example consider the function $y = f(x) = |x|$. For the value $x = 0$ we have

$$\Delta y = f(0 + \Delta x) - f(0) = f(\Delta x) = |\Delta x|.$$

If $\Delta x > 0$, then

$$\Delta y = \Delta x, \quad \lim_{\Delta x \to +0} \frac{\Delta y}{\Delta x} = 1.$$

If now $\Delta x < 0$, then

$$\Delta y = -\Delta x, \quad \lim_{\Delta x \to -0} \frac{\Delta y}{\Delta x} = -1.$$

The origin of the coordinate system is a sharp point of the graph of the function, which consists of the bisectors of the first and second quadrants.

87. Infinite derivatives. If the ratio of the increments $\Delta y/\Delta x$ tends to $+\infty$ or to $-\infty$ when $\Delta x \to 0$, this improper number is also called the derivative and denoted as before.

The geometric interpretation of the derivative as the angular coefficient of the tangent can be extended to this case; here, however, the tangent is parallel to the y-axis (Figs. 36a, b).

Similarly we can establish the concept of a one-sided infinite derivative. Incidentally, now the presence of one-sided infinite derivatives with different signs (Figs. 36c, d) implies the existence of a *unique* vertical tangent. The singularity of this case is due to the presence of a cusp directed vertically up or down.

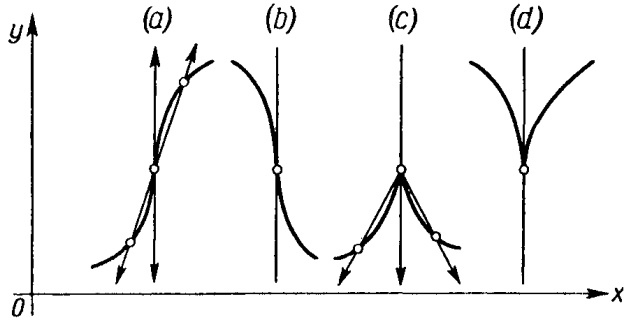

FIG. 36.

Suppose that for instance $f_1(x) = x^{1/3}$; for $x \neq 0$ formula 3 of Sec. 81 yields

$$f_1'(x) = \frac{1}{3} x^{-2/3} = \frac{1}{3x^{2/3}},$$

but it is not applicable for $x = 0$. At this point we calculate the derivative directly from its definition; construct the ratio

$$\frac{f_1(0+\Delta x) - f_1(0)}{\Delta x} = \frac{(\Delta x)^{1/3}}{\Delta x} = \frac{1}{(\Delta x)^{2/3}},$$

we observe that its limit when $\Delta x \to 0$ is $+\infty$. Similarly we find that for the function $f_2(x) = x^{2/3}$, for $x = 0$ the derivative from the left is $-\infty$ and from the right $+\infty$.

Making use of the extension of the concept of derivative we could complete the theorem of Sec. 80 on the derivative of the inverse function by the remark that when the function $f'(x_0)$ is equal to zero or $\pm\infty$, the derivative of the inverse function $g'(y_0)$ exists and is equal to $\pm\infty$ or zero, respectively. For instance, since the function $\sin x$ for $x = \pm\pi/2$ has the derivative $\cos(\pm\pi/2) = 0$, then for the

inverse function $\arcsin y$ for $y = \pm 1$ there exists an infinite derivative, namely $+\infty$.

88. Further examples of exceptional cases. *(1) Example of non-existence of the derivative.* The function $y = |x|$ at the point $x = 0$ [see Sec. 86] has no ordinary two-sided derivative. Even more interesting is the example of the function

$$f(x) = x \sin \frac{1}{x} \quad (\text{for } x \neq 0), \quad f(0) = 0,$$

continuous also for $x = 0$ [Sec. 67, *(4)*] but not having even one-sided derivatives at this point. In fact, the ratio

$$\frac{f(0+\Delta x) - f(0)}{\Delta x} = \frac{f(\Delta x)}{\Delta x} = \sin \frac{1}{\Delta x}$$

does not tend to any limit for $\Delta x \to \pm 0$.

The graph of this function (Fig. 21) clearly indicates that the chord OM_1 has no limiting position when M_1 tends to O; hence there is no tangent to the curve at the origin (even one-sided).

Subsequently we shall consider a remarkable example of a function continuous for all values of the argument, but having no derivative for any of them.

(2) Example of discontinuity of a function. If for the given function $y = f(x)$ there exists a finite derivative $y' = f'(x)$ at every point of an interval $\mathfrak{X}$, then this derivative is a function of x over $\mathfrak{X}$. In the numerous examples which we have so far considered this function turns out to be continuous. However, this is not always the case. Consider for instance the function

$$f(x) = x^2 \sin \frac{1}{x} \quad (\text{for } x \neq 0), \quad f(0) = 0.$$

If $x \neq 0$ the derivative can be calculated by the ordinary method

$$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x},$$

but the derived result is not valid for $x = 0$. Making use in this case of the definition of a derivative, we have

$$f'(0) = \lim_{\Delta x \to 0} \frac{f(0+\Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0} \Delta x \sin \frac{1}{\Delta x} = 0.$$

It is however clear that $f'(x)$ does not tend to any limit as $x \to 0$, and hence for $x = 0$ *the function $f'(x)$ has a discontinuity.*

In this example the discontinuity is of the second kind; later we shall see that the derivative cannot have any discontinuities of the first kind, i.e. jumps [Sec. 103].

§ 2. The differential

89. Definition of the differential. Consider the function $y = f(x)$ defined in an interval $\mathcal{X}$ and continuous at the point x_0. Then there corresponds to the increment Δx of the argument, the increment

$$\Delta y = \Delta f(x_0) = f(x_0 + \Delta x) - f(x_0),$$

which is infinitesimal if Δx is infinitesimal. The following problem is of great importance: to find whether there exists for Δy an infinitesimal $A \cdot \Delta x$ ($A = \text{const}$) linear in x, such that their difference compared with Δx is an infinitesimal of a higher order, i.e.

$$\Delta y = A \cdot \Delta x + o(\Delta x). \tag{1}$$

For $A \neq 0$ the validity of relation (1) indicates that the infinitesimal $A \cdot \Delta x$ is equivalent to the infinitesimal Δy and consequently it is the *principal part* of the latter, if the basic infinitesimal is Δx [Secs. 56, 57].

If the relation (1) holds, the function $y = f(x)$ is called *differentiable* (for the considered value $x = x_0$) and the expression $A \cdot \Delta x$ itself is called the *differential* of the function and is denoted by the symbol dy or $df(x_0)$. (In the latter case the particular value of x is indicated†.)

We state again that the differential of a function is described by two properties, namely: (*a*) it is a linear homogeneous function in the increment Δx of the argument and (*b*) it differs from the increment of the function by a quantity which is infinitesimal as $\Delta x \to 0$, of an order higher than Δx.

Let us examine some examples.

(1) The area Q of the circle of radius r is given by the formula $Q = \pi r^2$. If the radius r is increased by Δr, the corresponding increment ΔQ of Q is the area of the annulus bounded by the two concentric circles of radii r and $r + \Delta r$, respectively. The expression

$$\Delta Q = \pi(r + \Delta r)^2 - \pi r^2 = 2\pi r \cdot \Delta r + \pi(\Delta r)^2$$

shows immediately that the principal part of ΔQ, when $\Delta r \to 0$, is $2\pi r \cdot \Delta r$; this is exactly the differential dQ. It has a geometric meaning, namely it is the area of the rectangle (obtained by a "rectification" of the annulus) with the base equal to the length of the circle $2\pi r$, and height Δr.

† Here df as a whole symbol plays the role of a functional symbol.

(2) Consider now the free fall of a particle, governed by the law $s = gt^2/2$. In the time interval Δt, from t to $t+\Delta t$, the moving point covers the distance

$$\Delta s = \frac{g(t+\Delta t)^2}{2} - \frac{gt^2}{2} = gt \cdot \Delta t + \frac{g}{2}(\Delta t)^2.$$

As $\Delta t \to 0$ its principal part is $ds = gt \cdot \Delta t$. Let us recall that the velocity at the instant t is $v = gt$ [Sec. 76]; consequently we observe that the differential of the distance (which approximately replaces the increment of the distance) can be calculated as the distance covered by the point which in the course of the interval of time Δt would move with just this velocity.

90. The relation between the differentiability and the existence of the derivative. It is now easy to establish the validity of the following statement.

In order that the function $y = f(x)$ at the point x_0 should be differentiable, it is necessary and sufficient that there exists a finite derivative $y' = f'(x_0)$ for it at this point. If this condition is satisfied relation (1) *holds for the value of the constant A equal to this derivative:*

$$\Delta y = y'_x \cdot \Delta x + o(\Delta x). \tag{1a}$$

Necessity. If (1) holds, then

$$\frac{\Delta y}{\Delta x} = A + \frac{o(\Delta x)}{\Delta x},$$

hence, when Δx tends to zero we have in fact

$$A = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = y'_x.$$

Sufficiency follows immediately from Sec. 82, *(1)* [see (3a)].

Thus, the differential of the function $y = f(x)$ is always equal to the expression

$$dy = y'_x \cdot \Delta x^{\dagger}. \tag{2}$$

We emphasize that, in this expression, by Δx we understand an *arbitrary* increment of the independent variable, i.e. an arbitrary number (which it is frequently convenient to regard as independent of x). Moreover, it is by no means necessary to assume that Δx is infinites-

† It can easily be verified that this was just the way we constructed the differential in the examples examined in the preceding section. For instance, in the case *(1)*.

$$Q = \pi r^2, \quad Q'_r = 2\pi r, \quad dQ = 2\pi r \cdot \Delta r.$$

imal; but if $\Delta x \to 0$ the differential dy is also an infinitesimal, namely (when $y'_x \neq 0$) the principal part of the infinitesimal increment of the function Δy. This entitles us to set approximately

$$\Delta y \doteq dy \tag{3}$$

with increasing accuracy the smaller Δx becomes. We shall return to the approximate relation (3) in Sec. 93.

To interpret geometrically the differential dy and its connection with the increment Δy of the function $y = f(x)$ consider the graph of the function (Fig. 37). The values x of the argument and y of the function define point M on the curve. Draw at this point a tangent

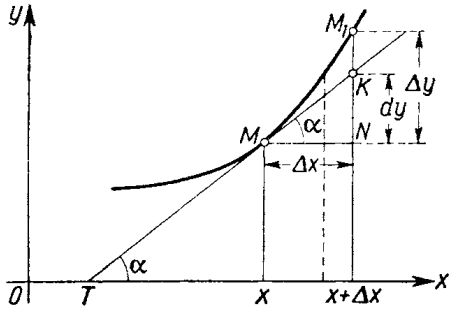

FIG. 37.

MT; we already know from Sec. 78 that its slope $\tan\alpha$ is equal to the derivative y'_x. If the abscissa x is increased by Δx, the ordinate of the curve y increases by $\Delta y = NM_1$. Furthermore, the ordinate of the tangent increases by NK. Computing NK as the side of the right-angled triangle MNK, we obtain

$$NK = MN \cdot \tan\alpha = y'_x \cdot \Delta x = dy.$$

Thus, Δy is the increment of the ordinate of the curve, while dy is the corresponding increment of the ordinate of the tangent.

Consider finally the independent variable x itself: by its differential we understand just the increment Δx, i.e. we agree to set

$$dx = \Delta x. \tag{4}$$

If the differential of the independent variable x is taken identical to the differential of the function $y = x$ (this is also a sort of convention), then from (2) we may prove formula (4) as follows:

$$dx = x'_x \cdot \Delta x = 1 \cdot \Delta x = \Delta x.$$

Taking into account the convention (4) we can now rewrite formula (2) defining the differential in the form

$$dy = y'_x dx. \tag{5}$$

This is the customary form.

Hence we obtain

$$y'_x = \frac{dy}{dx} \tag{6}$$

consequently the expression which was before regarded as a whole symbol can now be regarded as a fraction. The fact that on the left-hand side a fully determined number appears while on the right we have a ratio of two undetermined numbers dy and dx (in fact, $dx = \Delta x$ is arbitrary) should not confuse the reader; the numbers dy and dx are proportional, the derivative y'_x being the coefficient of proportionality.

The concept of the differential and the very term "the differential"† is due to Leibniz, who, however, did not give an exact definition of this concept. Besides differentials Leibniz also investigated differential quotients, i.e. quotients of two differentials, which is equivalent to our derivatives; however, for Leibniz the differential was the original concept. From the time of Cauchy who created the foundations of analysis by his theory of limits and who was the first to clearly define the derivative as a limit, it has been customary to begin by considering the derivative and then to construct the differential on the basis of the derivative.

91. Fundamental formulae and rules of differentiation. Computation of the differentials of functions is called the differentiation‡. Since the differential dy differs only by the factor dx from the derivative y'_x, from the list of derivatives of elementary functions [Sec. 81] it is easy to construct a list of their differentials:

1. $y = c$, $\quad dy = 0$;
2. $y = x^\mu$, $\quad dy = \mu x^{\mu-1} dx$;

$$y = \frac{1}{x}, \qquad dy = -\frac{dx}{x^2};$$

$$y = \sqrt{x}, \qquad dy = \frac{dx}{2\sqrt{x}};$$

† According to the Latin word *differentia* which means "difference".

‡ Incidentally, the same term is used to denote also the computation of derivatives.

3. $y = a^x$, $dy = a^x \log a\, dx$;
   $y = e^x$, $dy = e^x dx$;
4. $y = \log_a x$, $dy = \dfrac{\log_a e}{x} dx$;
   $y = \log x$, $dy = \dfrac{dx}{x}$;
5. $y = \sin x$, $dy = \cos x\, dx$;
6. $y = \cos x$, $dy = -\sin x\, dx$;
7. $y = \tan x$, $dy = \sec^2 x\, dx = \dfrac{dx}{\cos^2 x}$;
8. $y = \cot x$, $dy = \operatorname{cosec}^2 x dx = -\dfrac{dx}{\sin^2 x}$;
9. $y = \arcsin x$, $dy = \dfrac{dx}{\sqrt{(1-x^2)}}$;
10. $y = \arccos x$, $dy = -\dfrac{dx}{\sqrt{(1-x^2)}}$;
11. $y = \arctan x$, $dy = \dfrac{dx}{1+x^2}$;
12. $y = \operatorname{arc\,cot} x$, $dy = \dfrac{dx}{1+x^2}$.

The rules of differentiation† are the following:

I. $d(cu) = c\, du$,
II. $d(u \pm v) = du \pm dv$,
III. $d(uv) = v\, du + u\, dv$,
IV. $d\left(\dfrac{u}{v}\right) = \dfrac{v\, du - u\, dv}{v^2}$.

All the above formulae are easily derived from the corresponding rules for the derivatives. To prove, for instance, the last two:

$$d(uv) = (uv)'\, dx = (u'v + uv')\, dx$$
$$= v(u'\, dx) + u(v'\, dx) = v\, du + u\, dv,$$

† We now mean the computation of the differentials.

$$d\left(\frac{u}{v}\right) = \left(\frac{u}{v}\right)' dx = \frac{u'v - uv'}{v^2} dx$$

$$= \frac{v(u'\,dx) - u(v'\,dx)}{v^2} = \frac{v\,du - u\,dv}{v^2}.$$

92. Invariance of the form of the differential. The rule of differentiation of a compound function leads us to a remarkable and important property of the differential.

Suppose that the functions $y = f(x)$ and $x = \varphi(t)$ are such that the compound function $y = f(\varphi(t))$ can be constructed. If the derivatives y'_x and x'_t exist, then according to rule V [Sec. 84] there also exists the derivative

$$y'_t = y'_x x'_t. \tag{7}$$

If x is regarded as the independent variable, the differential dy is given by formula (5). We now pass to the independent variable t; then we have another expression for the differential, namely

$$dy = y'_t\,dt.$$

However, replacing the derivative y'_t by its expression (7) and noting that $x'_t\,dt$ is the differential of x treated as a function of t, we finally obtain

$$dy = y'_x x'_t\,dt = y'_x\,dx,$$

i.e. we return to the former form of the differential.

Thus, we observe that the form of the differential can be preserved even if the former independent variable is replaced by a new one. We may always write the differential of y in the form (5) whether x is the independent variable or not; the only difference is that if t is taken to be the independent variable, dx denotes not an arbitrary increment Δx but the differential of x as a function of t. This property is called the invariance of the form of the differential.

Since formula (5) yields directly formula (6) expressing the derivative y'_x by the differentials dx and dy, the latter formula also remains valid, *no matter with respect to which independent variable (of course, the same in both cases) the considered differentials are computed.*

Suppose, for instance, that $y = \sqrt{(1-x^2)}$ $(-1 < x < 1)$; thus

$$y'_x = -\frac{x}{\sqrt{(1-x^2)}}.$$

Now set $x = \sin t (-\pi/2 < t < \pi/2)$. Then $y = \sqrt{(1-\sin^2 t)} = \cos t$ and $dx = \cos t \cdot dt$, $dy = -\sin t \cdot dt$. It can easily be verified that formula (6) represents only another expression for the derivative computed above.

Remark. The possibility of expressing the derivative by the differentials taken with respect to any variable leads in particular to the fact that the formulae

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}, \quad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx},$$

expressing (in Leibniz notation) the rules of differentiation of the inverse and compound functions, become simple algebraic identities (since all differentials can here be taken with respect to the same variable). Incidentally the reader should not think that this constitutes a new derivation of the considered formulae; first of all the *existence* of the derivatives on the left is not proved here; the main fact, however, is that we have used the invariance of the form of the differential which is itself a result of rule V.

93. Differentials as a source of approximate formulae. We have found that as $\Delta x \to 0$ the differential dy of the function y (provided $y'_x \neq 0$) represents the principal part of the infinitesimal increment of the function, Δy. Thus, $\Delta y \sim dy$, whence

$$\Delta y = dy, \tag{3}$$

or in more detail

$$\Delta f(x_0) = f(x_0 + \Delta x) - f(x_0) \doteq f'(x_0)\Delta x \tag{3a}$$

with accuracy to an infinitesimal of an order higher than Δx. This means that [Sec. 56] the relative error of this relation becomes arbitrarily small for a sufficiently small Δx.

This fact also follows directly from Fig. 37 which represents the geometric interpretation of the differential. It is seen from the graph that when Δx decreases we can with increasing relative accuracy replace the increment of the ordinate of the curve by the increment of the ordinate of the tangent.

The convenience of replacing the increment of the function Δy by its differential dy arises from the fact that dy depends linearly on Δx, while Δy is usually a more complicated function of Δx.

If we set $\Delta x = x - x_0$ so $x_0 + \Delta x = x$, the relation (3a) takes the form

$$f(x) - f(x_0) \doteq f'(x_0)(x - x_0)$$

or

$$f(x) \doteq f(x_0) + f'(x_0)(x - x_0).$$

For values of x close to x_0 the function $f(x)$, in accordance with this formula, can approximately be replaced by a linear function. Geometrically this corresponds to replacing the section of the curve $y = f(x)$ adjacent to the point $(x_0, f(x_0))$ by a section of the tangent to the curve at this point:

$$y = f(x_0) + f'(x_0)(x - x_0)\dagger$$

(see Fig. 37). Taking for simplicity $x_0 = 0$ and confining ourselves to small values of x we have the approximate formula

$$f(x) \doteq f(0) + f'(0)x.$$

Consequently, replacing $f(x)$ by various elementary functions it is easy to derive a number of formulae:

$$(1+x)^\mu \doteq 1 + \mu x, \quad \text{in particular,} \quad \sqrt{(1+x)} \doteq 1 + \frac{1}{2}x,$$

$$e^x \doteq 1 + x, \quad \log(1+x) \doteq x, \quad \sin x \doteq x, \quad \tan x \doteq x, \text{ etc.},$$

some of which we already know.

94. Application of differentials in estimating errors. It is particularly convenient and natural to employ the concept of the differential to estimate the error in approximate calculations. Suppose, for instance, that we measure or calculate directly a quantity x, while a quantity y depending on it is determined from the formula $y = f(x)$. In measuring the quantity x we may make an error Δx which results in an error Δy for y. Since the magnitude of the error is small we may set

$$\Delta y = y'_x \Delta x,$$

i.e. we replace the increment by the differential. Let δx be the maximum absolute error of quantity x (in ordinary circumstances this bound of the error in making the measurement is known). It is then evident that we may take for the maximum absolute error (the bound of the error) for y, the quantity

$$\delta y = |y'_x| \delta x. \tag{8}$$

(1) Suppose, for instance, that to determine a volume of a sphere we first (by means of a micrometer, a device for measuring thickness, etc.) directly measure the diameter D of the sphere and compute the volume V by means of the formula

$$V = \frac{\pi}{6} D^3.$$

---

† In fact, the equation of the straight line with slope k passing through point (x_0, y_0) is

$$y = y_0 + k(x - x_0);$$

in the case of the tangent we set $y_0 = f(x_0)$, $k = f'(x_0)$.

Since $V'_D = (\pi/2)\,D^2$ we have, in view of (8), in this case

$$\delta V = \frac{\pi}{2} D^2 \delta D.$$

Dividing this relation by the preceding one we obtain

$$\frac{\delta V}{V} = 3\,\frac{\delta D}{D},$$

and hence the (maximum) relative error of the calculated volume is three times greater than the (maximum) relative error of the measured magnitude of the diameter.

(2) If the number x for which we calculate the decimal logarithm $y = \log_{10} x$ contains an error, it influences the logarithm, which will also contain an error.

Now $y'_x = M/x$ ($M \doteq 0.4343$) and hence, by virtue of formula (8)

$$\delta y = 0.4343\,\frac{\delta x}{x}.$$

Thus, the (maximum) absolute error of the logarithm can be determined in terms of the (maximum) relative error of the number, and conversely.

This result has numerous applications. For instance, it can be used to give an approximate value for the accuracy of the ordinary slide rule with the scale of 25 cm = 250 mm. If in setting the slide we make an error of, for instance, 0.1 mm in both directions, there arises an error in the logarithm of

$$\delta y = \frac{0.1}{250} = 0.0004.$$

Hence, in accordance with our formula

$$\frac{\delta x}{x} = \frac{0.0004}{0.4343} \doteq 0.001.$$

Nevertheless the relative error in all parts of the scale is the same.

§ 3. Derivatives and differentials of higher orders

95. Definition of derivatives of higher orders. If a function $y = f(x)$ has a finite derivative $y' = f'(x)$ in an interval $\mathcal{X}$, then the latter is a new function of x and it is possible that in turn this function may have a derivative at a point x of $\mathcal{X}$, finite or otherwise. If it does, it is called *the derivative of the second order* or *the second derivative* of the function $y = f(x)$ at the point considered; it is denoted by one of the following symbols:

$$\frac{d^2y}{dx^2},\quad y'',\quad D^2y;\quad \frac{d^2f(x_0)}{dx^2},\quad f''(x_0),\quad D^2f(x_0).$$

For instance, we found in Sec. 78 that the velocity v of the motion of a point is equal to the derivative of the distance covered by the point with respect to the time t, $v = ds/dt$, while the acceleration a is the derivative of the velocity v with respect to time, $a = dv/dt$. Consequently, the acceleration is called the second derivative of the distance with respect to time, $a = d^2s/dt^2$.

Similarly, if the function $y = f(x)$ has a finite second derivative in the whole interval $\mathcal{X}$, i.e. at every point of this interval, then its derivative, finite or otherwise, at a point x_0 from $\mathcal{X}$ is called the *derivative of the third order* or the *third derivative* of the function $y = f(x)$ at this point, and it is denoted as follows:

$$\frac{d^3y}{dx^3},\ y''',\ D^3y;\qquad \frac{d^3f(x_0)}{dx^3},\ f'''(x_0),\ D^3f(x_0).$$

In a similar manner we pass from the third derivative to the fourth, and so on. If we assume that the concept of the $(n-1)$th derivative has already been defined, and the latter exists and is finite in the whole interval $\mathcal{X}$, then its derivative at a point x_0 of this interval is called the *derivative of n-th order* or *n-th derivative* of the original function $y = f(x)$; the following notation is used for this:

$$\frac{d^ny}{dx^n},\ y^{(n)},\ D^ny;\qquad \frac{d^nf(x_0)}{dx^n},\ f^{(n)}(x_0),\ D^nf(x_0).$$

Sometimes, when the notations of Lagrange or Cauchy are used, it may be necessary to indicate the variable with respect to which the derivative is taken; then it is shown as a suffix,

$$y''_{x^2},\ D^3_{x^3}f(x),\ f^{(n)}_{x^n}(x_0),\ \text{etc.},$$

$x^2, x^3, \ldots$ being conventionally written instead of $xx, xxx, \ldots$. For instance, we may write $a = s''_{t^2}$.

(It should be clear to the reader that all of the symbols

$$\frac{d^nf}{dx^n},\ f^{(n)} \text{ or } f^{(n)}_{x^n},\ D^nf \text{ or } D^n_{x^n}f$$

can be regarded as functional symbols.)

Thus, we have defined the concept of the nth derivative by induction, passing successively from the first derivative to the higher ones. The relation defining the nth derivative

$$y^{(n)} = [y^{(n-1)}]'$$

is also called a recurrence relation, for it relates the nth derivative to the $(n-1)$th.

The calculation of derivatives of the nth order itself, for a given number n, is carried out in accordance with the rules which have already been described.

For instance, if

$$y = \tfrac{1}{2}x^4 - \tfrac{1}{6}x^3 + 2x^2 + \tfrac{4}{3}x - \tfrac{1}{2},$$

then

$$y' = 2x^3 - \tfrac{1}{2}x^2 + 4x + \tfrac{4}{3}, \quad y'' = 6x^2 - x + 4,$$

$$y''' = 12x - 1, \quad y'''' = 12,$$

hence all subsequent derivatives vanish identically. If now

$$y = \log[x + \sqrt{(x^2+1)}],$$

then

$$y' = \frac{1}{\sqrt{(x^2+1)}}, \quad y'' = -\frac{x}{(x^2+1)^{3/2}}, \quad y''' = \frac{2x^2-1}{(x^2+1)^{5/2}}, \quad \text{etc.}$$

Observe that for the derivatives of higher orders we can also establish by induction the concept of one-sided derivatives [cf. Sec. 86]. If the function $y = f(x)$ is defined in an interval $\mathfrak{X}$ only, then speaking of a derivative of an arbitrary order at an end-point we always mean a one-sided derivative.

96. General formulae for derivatives of arbitrary order. Thus, in order to calculate the nth derivative of a function it is in general necessary to calculate first the derivatives of all preceding orders. However, in some cases it is possible to find a general formula for the nth derivative which depends directly on n and does not contain any of the symbols of the preceding derivatives.

In deriving such general relations it is sometimes useful to employ the formulae

$$(cu)^{(n)} = cu^{(n)}, \quad (u \pm v)^{(n)} = u^{(n)} \pm v^{(n)},$$

extending to the case of higher derivatives the familiar rules I and II of Sec. 83. They can easily be deduced by a successive application of these rules.

(1) Consider first the power function $y = x^\mu$ where μ is an arbitrary real number. We have

$$y' = \mu x^{\mu-1}, \quad y'' = \mu(\mu-1)x^{\mu-2},$$
$$y''' = \mu(\mu-1)(\mu-2)x^{\mu-3}, \ldots.$$

Hence we easily derive the general rule

$$y^{(n)} = \mu(\mu-1)\ldots(\mu-n+1)x^{\mu-n},$$

which can be proved by the method of mathematical induction.

Taking, for instance, $\mu = -1$ we obtain

$$\left(\frac{1}{x}\right)^{(n)} = (-1)(-2)\ldots(-n)x^{-1-n} = \frac{(-1)^n \cdot n!}{x^{n+1}}.$$

When μ itself is a positive integer m, then the mth derivative of x^m is already a constant number $m!$ and all subsequent derivatives are zero. It follows that the same is true for a polynomial of degree m.

(2) Suppose now that $y = \log x$. First of all we have

$$y' = (\log x)' = \frac{1}{x}.$$

Using (1), where $\mu = -1$, and n is replaced by $n-1$, we obtain

$$y^{(n)} = (y')^{(n-1)} = \left(\frac{1}{x}\right)^{(n-1)} = \frac{(-1)^{n-1}(n-1)!}{x^n}.$$

(3) If $y = a^x$ we have

$$y' = a^x \log a, \quad y'' = a^x(\log a)^2, \ldots.$$

The general formula

$$y^{(n)} = a^x (\log a)^n$$

can easily be proved by the method of mathematical induction.

It is evident, in particular, that

$$(e^x)^{(n)} = e^x.$$

(4) Let $y = \sin x$; then

$$y' = \cos x, \quad y'' = -\sin x, \quad y''' = -\cos x,$$
$$y'''' = \sin x, \quad y^{(5)} = \cos x, \ldots.$$

It is difficult to find a general expression for the nth derivative in this manner. But the procedure is at once simplified if we rewrite the

formula for the first derivative in the form $y' = \sin(x+\pi/2)$; it becomes evident that in each differentiation the number $\pi/2$ is added to the argument. Hence

$$(\sin x)^{(n)} = \sin\left(x + n\frac{\pi}{2}\right).$$

In an analogous way we obtain the formula

$$(\cos x)^{(n)} = \cos\left(x + n\frac{\pi}{2}\right).$$

(5) We now examine the function $y = \arctan x$. Let us attempt to express $y^{(n)}$ by y. Since $x = \tan y$ we have

$$y' = \frac{1}{1+x^2} = \cos^2 y = \cos y \cdot \sin\left(y + \frac{\pi}{2}\right).$$

Differentiating again with respect to x (and remembering that y is a function of x) we obtain

$$y'' = \left[-\sin y \cdot \sin\left(y + \frac{\pi}{2}\right) + \cos y \cdot \cos\left(y + \frac{\pi}{2}\right)\right] \cdot y'$$

$$= \cos^2 y \cdot \cos\left(2y + \frac{\pi}{2}\right) = \cos^2 y \cdot \sin 2\left(y + \frac{\pi}{2}\right).$$

The next differentiation yields

$$y''' = \left[-2\sin y \cdot \cos y \cdot \sin 2\left(y + \frac{\pi}{2}\right) + 2\cos^2 y \cdot \cos 2\left(y + \frac{\pi}{2}\right)\right] \cdot y'$$

$$= 2\cos^3 y \cdot \cos\left(3y + 2 \cdot \frac{\pi}{2}\right) = 2\cos^3 y \cdot \sin 3\left(y + \frac{\pi}{2}\right).$$

The general formula

$$y^{(n)} = (n-1)!\cos^n y \cdot \sin n\left(y + \frac{\pi}{2}\right)$$

can be proved by means of mathematical induction.

97. The Leibniz formula. We observed at the beginning of the preceding section that rules I and II of Sec. 83 can be extended directly to the case of derivatives of arbitrary orders. The case is more complicated with rule III concerning the differentiation of a product.

Assume that the functions u, v of x each have derivatives up to the nth order (inclusive); we now prove that then the product $y = uv$ also has an nth derivative, and we shall find an expression for it.

Applying rule III let us successively differentiate the product; thus we find that

$$y' = u'v + uv', \quad y'' = u''v + 2u'v' + uv'',$$
$$y''' = u'''v + 3u''v' + 3u'v'' + uv''', \ \ldots.$$

It is easy to see a rule from which all the above formulae can be constructed; the right sides resemble the expansion of a power of binomial: $u+v$, $(u+v)^2$, $(u+v)^3$, ..., but the powers of u, v are replaced by the derivatives of the corresponding orders. The resemblance is even more marked if we write in the deduced formulae $u^{(0)}, v^{(0)}$ instead of u, v. Extending this law to an arbitrary n we arrive at the general formula†

$$y^{(n)} = (uv)^{(n)} = \sum_{i=0}^{n} C_n^i u^{(n-i)} v^{(i)}$$
$$= u^{(n)}v + nu^{(n-1)}v' + \frac{n(n-1)}{1\cdot 2} u^{(n-2)}v'' + \ldots$$
$$+ \frac{n(n-1)\ldots(n-i+1)}{1\cdot 2\ldots i} u^{(n-i)}v^{(i)} + \ldots + uv^{(n)}. \qquad (1)$$

To prove its validity we again use the method of mathematical induction. Assume that it holds for some value of n. If for the functions u, v there the $(n+1)$th derivatives also exist, (1) can be differentiated once more with respect to x; we obtain

$$y^{(n+1)} = \sum_{i=0}^{n} C_n^i [u^{(n-i)}v^{(i)}]' = \sum_{i=0}^{n} C_n^i u^{(n-i+1)}v^{(i)} + \sum_{i=0}^{n} C_n^i u^{(n-i)}v^{(i+1)}.$$

† The symbol Σ denotes the sum of terms of one type. When the terms depend on one index ranging over a definite range the appropriate bounds are indicated (below and above the sign Σ). For instance,

$$\sum_{i=0}^{n} a_i = a_0 + a_1 + \ldots + a_n,$$

$$\sum_{k=1}^{m} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{m}, \quad \text{etc.}$$

Now collect the terms of the two last sums; these contain the same products of the functions u and v (it is readily seen that the sum of the orders in this product is $n+1$). The product $u^{(n+1)}v^{(0)}$ enters only the first sum (for $i=0$); its coefficient in this sum is $C_n^0=1$. Analogously $u^{(0)}v^{(n+1)}$ enters only the second sum (in the term with the number $i=n$), the coefficient being $C_n^n=1$. All remaining products entering these sums are of the form $u^{(n+1-k)}v^{(k)}$, and $1\leqslant k\leqslant n$. Every such product is encountered both in the first sum (the term with the number $i=k$) and in the second sum (the term with the number $i=k-1$). The sum of the corresponding coefficients is $C_n^k+C_n^{k-1}$. However, it is known that

$$C_n^k+C_n^{k-1}=C_{n+1}^k.$$

Consequently we finally have that

$$y^{(n+1)}=u^{(n+1)}v^{(0)}+\sum_{k=1}^{n}C_{n+1}^k u^{[(n+1)-k]}v^{(k)}+u^{(0)}v^{(n+1)}$$

$$=\sum_{k=0}^{n+1}C_{n+1}^k u^{[(n+1)-k]}v^{(k)},$$

for

$$C_{n+1}^0=C_{n+1}^{n+1}=1.$$

We have derived for $y^{(n+1)}$ an expression entirely analogous to (1) (n being replaced by $n+1$); this completes the proof of formula (1) for all positive integers n.

The established formula is called *the Leibniz formula.* It is frequently useful in deducing general expressions for the nth derivative.

Observe that the same formula could be established for the nth derivative of a product of several functions, $y=uv\ldots t$; it is similar to the expansion of the polynomial $(u+v+\ldots+t)^n$.

Example. We proceed to find the general expression for the nth derivative of the function

$$y=e^{ax}\cdot\sin bx.$$

By the Leibniz formula we have

$$y^{(n)}=e^{ax}\cdot a^n\cdot\sin bx+ne^{ax}\cdot a^{n-1}b\cdot\cos bx-\frac{n(n-1)}{1\cdot 2}e^{ax}\cdot a^{n-2}b^2\cdot\sin bx$$

$$-\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}e^{ax}\cdot a^{n-3}b^3\cdot\cos bx+\ldots$$

or

$$y^{(n)} = e^{ax}\left\{\sin bx\left[a^n - \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \ldots\right] + \cos bx\left[na^{n-1}b - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \ldots\right]\right\}.$$

98. Differentials of higher orders. We now consider differentials of higher orders; they are also determined by induction. By the *differential of the second order or the second differential* of the function $y = f(x)$ at a point, we mean the differential of its (first) differential at this point. Symbolically

$$d^2y = d(dy).$$

By *the differential of the third order or the third differential* we understand the differential of the second differential

$$d^3y = d(d^2y).$$

Generally, by the *differential of n-th order* or *n-th differential* of the function $y = f(x)$ we understand the differential of its $(n-1)$th differential, i.e.

$$d^ny = d(d^{n-1}y).$$

If we make use of the functional notation, the successive differentials may be denoted as follows:

$$d^2f(x_0),\ d^3f(x_0),\ \ldots,\ d^nf(x_0),\ \ldots,$$

and we can indicate the particular value of $x = x_0$ at which the differentials are to be taken.

In computing differentials of higher orders it is very important to remember that dx is an arbitrary number, *independent* of x, which in the differentiation with respect to x should be regarded as a constant factor. Thus we have (assuming all the time that the derivatives of the required orders exist)

$$d^2y = d(dy) = d(y'dx) = dy'dx = (y''dx)\,dx = y''dx^2,$$

$$d^3y = d(d^2y) = d(y''dx) = dy''dx = (y'''dx)\,dx^2 = y'''dx^3\dagger,$$

† By dx^2, dx^3, etc., we always understand the powers of the differential, i.e. $(dx)^2$, $(dx)^3$, ... The differential of a power is always indicated as follows: $d(x^2)$, $d(x^3)$, ...

etc. We can easily derive the general law

$$d^n y = y^{(n)} dx^n \tag{2}$$

and prove it by mathematical induction. It implies that

$$y^{(n)} = \frac{d^n y}{dx^n},$$

so that henceforth we may regard this symbol as a fraction.

Making use of relation (2) it is now easy to transform the Leibniz formula to differentials. It is sufficient to multiply it throughout by dx^n to obtain

$$d^n(uv) = \sum_{i=0}^{n} C_n^i d^{n-i} u d^i v \qquad (d^0 u = u,\ d^0 v = v).$$

Leibniz himself deduced this formula in exactly this form.

99. Violation of the invariance of the form for differentials of higher orders. Remembering that the first differential of a function possesses the property of invariance of the form, it is natural to consider the question as to whether differentials of higher orders also have this property. We shall prove, for instance, that already the second differential does not have this property.

Thus, suppose that $y = f(x)$ and $x = \varphi(t)$ so that y may be regarded as a compound function of t, i.e. $y = f(\varphi(t))$. Its (first) differential with respect to t can be written in the form $dy = y'_x dx$ where $dx = x'_t dt$ is a function of t. The second differential with respect to t is:

$$d^2 y = d(y'_x dx) = dy'_x dx + y'_x d(dx).$$

Again making use of the invariance of the form of the first differential we can write the differential dy'_x in the form $dy'_x = y''_{x^2} dx$, and hence, finally

$$d^2 y = y''_{x^2} dx^2 + y'_x d^2 x, \tag{3}$$

while when x is regarded as the independent variable, the second differential would have the form $d^2 y = y''_{x^2} dx^2$. Of course, expression (3) for $d^2 y$ is more general; if, in particular, x is the independent variable, then $d^2 x = 0$ and only the first term remains.

Consider an example. Suppose that $y = x^2$; since x is the independent variable

$$dy = 2x\,dx, \quad d^2y = 2dx^2.$$

Now set $x = t^2$; then $y = t^4$ and

$$dy = 4t^3\,dt, \quad d^2y = 12t^2\,dt^2.$$

The new expression for dy can also be derived from the former one if we set in the latter $x = t^2$, $dx = 2t\,dt$. The case is different for d^2y: making this substitution we obtain $8t^2\,dt^2$ instead of $12t^2\,dt^2$!

Formula (3) in this case has the form

$$d^2y = 2dx^2 + 2x\,d^2x.$$

Substituting here $x = t^2$, $dx = 2t\,dt$, $d^2x = 2dt^2$ we obtain the correct result $12t^2\,dt^2$.

Thus, if x is no longer the independent variable the differential of second order d^2y is expressed by the differentials of x by the two-term formula (3). For the differentials of third and higher orders the number of additional terms (in passing to the new independent variable) increases even more. Accordingly, in expressions of the higher derivatives y''_{x^2}, y'''_{x^3}, ... by the differentials

$$y''_{x^2} = \frac{d^2y}{dx^2}, \quad y'''_{x^3} = \frac{d^3y}{dx^3}, \quad \ldots \tag{4}$$

it is not possible to take the differentials with respect to any variable; the variable x must be used.

CHAPTER 6

BASIC THEOREMS OF DIFFERENTIAL CALCULUS

§ 1. Mean value theorems

100. Fermat's theorem. The knowledge of the derivative (or a number of derivatives) of a function makes it possible to draw conclusions regarding the function itself. The basis of various applications of the concept of a derivative (see Chapters 7 and 13) rests on certain simple but important theorems and formulae to which this chapter is devoted.

We begin by examining a statement which is linked with the name of Fermat†. Of course, he did not announce it in the form in which we present it here (Fermat did not know of the concept of a derivative); however, our form re-establishes the essence of Fermat's device as applied by him to determining the greatest and the smallest values of a function (see Chapter 14).

FERMAT'S THEOREM. *Suppose that a function $f(x)$ is defined in an interval $\mathcal{X}$ and that it takes at an interior point of the interval its greatest (smallest) value. If at this point there exists the finite derivative $f'(c)$, then, necessarily, $f'(c) = 0$.*

Proof. For definiteness let $f(x)$ take at a point c its greatest value; hence for all x from $\mathcal{X}$ we have

$$f(x) \leqslant f(c).$$

According to the definition of the derivative

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c},$$

† Pierre Fermat (1601–1665)—an outstanding French mathematician whose name is closely connected with the early history of the analysis of infinitesimals (see Chapter 14).

this limit being independent of the approach of x to c from the left or from the right. But for $x > c$ the expression

$$\frac{f(x)-f(c)}{x-c} \leqslant 0,$$

and hence passing to the limit, $x \to c+0$, we obtain

$$f'(c) \leqslant 0. \tag{1}$$

If now $x < c$, then

$$\frac{f(x)-f(c)}{x-c} \geqslant 0,$$

passing to the limit, $x \to c-0$, we have

$$f'(c) = 0. \tag{2}$$

Comparing relations (1) and (2) we arrive at the required result

$$f'(c) = 0.$$

Remark. The reasoning carried out above proves, essentially, that at the considered point c an infinite (two-sided) derivative cannot exist. Thus, the statement of the theorem is not altered if we assume, at the considered point, the existence of a (two-sided) derivative, without stating beforehand that it has to be finite.

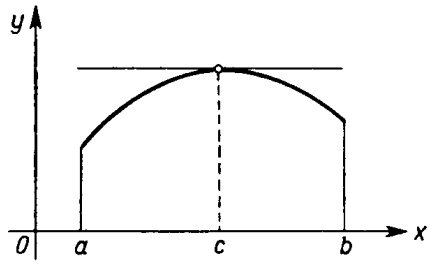

FIG. 38.

Let us recall [Secs. 77, 78] the geometric interpretation of the derivative $y' = f'(x)$ as the slope of the tangent to the curve $y = f(x)$; the vanishing of the derivative $f'(c)$ means, geometrically, that at the considered point of the curve the tangent is parallel to the x-axis. Figure 38 clearly illustrates this statement.

It was essential in the proof to make use of the assumption that c is an interior point of the interval, since otherwise we would have

to take into account points x on the right of c and points x on the left of it. Without this assumption the theorem would no longer be true: if a function $f(x)$ is defined in a closed interval and reaches its greatest (smallest) value at one of the ends of the interval, then the derivative $f'(x)$ may not vanish (if it exists) at this end. We leave it to the reader to find an appropriate example.

101. Rolle's theorem. Numerous theorems and formulae of the differential calculus and its applications are based on the following simple but important theorem attributed to Rolle†.

ROLLE'S THEOREM. *Suppose that (1) the function $f(x)$ is defined and is continuous in the closed interval $[a, b]$, (2) there exists a finite derivative $f'(x)$, at least in the open interval (a, b)‡, (3) at the ends of the interval the function takes equal values, i.e. $f(a) = f(b)$.*

Under such circumstances, between a and b a point c can be found $(a < c < b)$, such that $f'(c) = 0$.

Proof. $f(x)$ is continuous in the closed interval $[a, b]$ and consequently by Weierstrass' second theorem [Sec. 73] it attains both its greatest value M and its smallest value m in this interval.

Consider two cases.

1. $M = m$. Then $f(x)$ has a constant value in the interval $[a, b]$; in fact, the inequality $m \leqslant f(x) \leqslant M$ in this case yields $f(x) = M$ for all x; hence $f'(x) = 0$ in the whole interval, and for c we may take any point in (a, b).

2. $M > m$. We know that both these values of the function are attained, but since $f(a) = f(b)$ they cannot *both* occur on the boundaries of the interval, and at least one of them occurs at a point c *between* a and b. Therefore it follows from Fermat's theorem that the derivative $f'(c)$ vanishes at this point. This completes the proof of the theorem.

In the language of geometry Rolle's theorem states the following: if the extreme ordinates of the curve $y = f(x)$ are equal, a point

† Michel Rolle (1652–1719)—a French mathematician who for a long time was opposed to the new calculus and adopted it only at the end of his life. This theorem was announced by him for polynomials.

‡ Obviously, the continuity of the function $f(x)$ in (a, b) follows from (2), but neither here nor later shall we attempt to split up the condition of the theorem into independent assumptions.

can be found on the curve at which the tangent is parallel to the x-axis (Fig. 39).

We emphasize that the continuity of the function $f(x)$ in the *closed* interval $[a, b]$ and the existence of the derivative in the *whole* open interval (a, b) are essential for the validity of the theorem. The function $f(x) = x - E(x)$ satisfies in the interval [0, 1] all conditions of the theorem, except that it has a discontinuity at $x = 1$, and the derivative $f'(x) = 1$, everywhere in (0, 1). The function defined by the relations $f(x) = x$ for $0 \leqslant x \leqslant 1/2$ and $f(x) = 1 - x$ for $1/2 < x < 1$ also satisfies all conditions in the considered interval, except that at $x = 1/2$ a finite (two-sided) derivative does not exist; now in the left half of the interval $f'(x) = +1$ while in the right one $f'(x) = -1$.

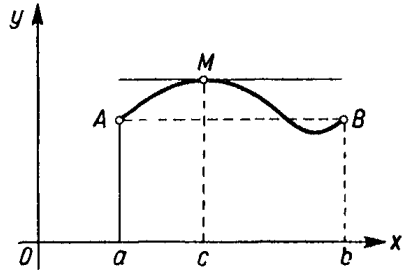

FIG. 39.

Similarly, condition (3) of the theorem is important: the function $f(x) = x$ in the interval [0, 1] satisfies all conditions of the theorem except the third, and everywhere its derivative $f'(x) = 1$.

The construction of the appropriate diagrams is left to the reader.

102. Theorem on finite increments. We now consider the direct consequences of Rolle's theorem. The first is the following theorem on finite increments announced by Lagrange.

LAGRANGE'S THEOREM. *Suppose that (1) $f(x)$ is defined and is continuous in the closed interval $[a, b]$, (2) there exists a finite derivative $f'(x)$ at least in the open interval (a, b). Then between a and b a point c can be found $(a < c < b)$, such that at this point the following relation holds::*

$$\frac{f(b) - f(a)}{b - a} = f'(c). \tag{3}$$

Proof. Introduce an auxiliary function defined in the interval $[a, b]$ by means of the relation

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

It satisfies all conditions of Rolle's theorem. In fact, it is continuous in $[a, b]$ since it represents a difference between the continuous function $f(x)$ and a linear function. In the interval (a, b) it has a definite finite derivative

$$F'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

Finally a direct substitution proves that $F(a) = F(b)$, i.e. $F(x)$ takes equal values on the ends of the interval.

Consequently we may apply Rolle's theorem to the function $F(x)$ and hence there exists a point c in (a, b) where $F'(c) = 0$. Consequently

$$f'(c) - \frac{f(b) - f(a)}{b - a} = 0,$$

whence

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

This completes the proof.

Rolle's theorem constitutes a particular case of Lagrange's theorem; the remarks made above, which concern conditions (1) and (2) of the theorem, remain valid in this case as well.

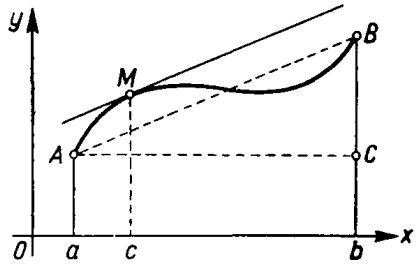

FIG. 40.

Proceeding to the geometric interpretation of Lagrange's theorem (Fig. 40) we observe that the ratio

$$\frac{f(b) - f(a)}{b - a} = \frac{CB}{AC}$$

is the slope of the chord AB and $f'(c)$ is the slope of the tangent to the curve $y = f(x)$ at the point with abscissa $x = c$. Thus, the

statement of Lagrange's theorem is equivalent to the following: on the arc AB there always exists at least one point M at which the tangent is parallel to the chord AB.

The formula

$$\frac{f(b)-f(a)}{b-a}=f'(c) \quad \text{or} \quad f(b)-f(a)=f'(c)(b-a)$$

is called *Lagrange's formula* or *the formula of finite increments.* It is evident that it also holds for the case $a>b$.

Now take an arbitrary value x_0 in the interval $[a, b]$ and increase it by $\Delta x \gtrless 0$, such that $x_0+\Delta x$ remains within the interval. We apply Lagrange's formula to the interval $[x_0, x_0+\Delta x]$ for $\Delta x>0$, or to the interval $[x_0+\Delta x, x_0]$ for $\Delta x<0$. Then Lagrange's formula takes the form

$$\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}=f'(c) \tag{3a}$$

or

$$\Delta f(x_0)=f(x_0+\Delta x)-f(x_0)=f'(c)\Delta x. \tag{4}$$

The number c lying between x_0 and $x_0+\Delta x$ in our case can be represented as follows:

$$c=x_0+\theta\,\Delta x, \quad \text{where} \quad 0<\theta<1^{\dagger}.$$

This relation giving the exact expression for the increment of the function in an arbitrary finite increment Δx of the argument should be compared with the approximate relation [Sec. 93, (3a)]

$$\Delta f(x_0)=f(x_0+\Delta x)-f(x_0)\doteq f'(x_0)\Delta x,$$

the relative error of which tends to zero only for an infinitesimal Δx. Hence we have the words "finite increments" in the name of the formula (and the theorem).

An inconvenience of Lagrange's formula is caused by the appearance of the unknown number c (or θ) in it[‡]; however, this formula is still of considerable use in analysis.

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† Sometimes it is said that θ is "the regular fraction"; one should not, however, think that a rational fraction is meant, for the number θ may also turn out to be irrational.

‡ Only in a few cases can we find it; for instance, for the quadratic function $f(x)=ax^2+bx+c$ it is easy to verify that $\theta=1/2$.

103. The limit of the derivative. A useful example of such an application arises from the following remark. Assume that the function $f(x)$ is continuous in the interval $[x_0, x_0 + H]$ $(H > 0)$ and has a finite derivative $f'(x)$ for $x > x_0$. If the following limit exists (finite or otherwise)

$$\lim_{x \to x_0 + 0} f'(x) = K,$$

the same value is taken by the derivative from the right at the point x_0. In fact, for $0 < \Delta x \leqslant H$ relation (3a) holds. Since the argument c of the derivative lies between x_0 and $x_0 + \Delta x$, for $\Delta x \to 0$ it tends to x_0, and consequently the right-hand side of the relation, and hence the left-hand side, tend to the limit K; this is what was to be proved. An analogous proposition can be established for the left-hand vicinity of the point x_0.

Consider as an example the function

$$f(x) = x \arcsin x + \sqrt{(1 - x^2)}$$

defined over the interval $[-1, 1]$. If $-1 < x < 1$, by the ordinary rules of differential calculus we easily find that

$$f'(x) = \arcsin x.$$

As $x \to 1 - 0$ $(x \to -1 + 0)$ it is evident that this derivative tends to the limit $\pi/2$ $(-\pi/2)$; hence at $x = \pm 1$ there exist (one-sided) derivatives $f'(\pm 1) = \pm \pi/2$.

Returning to the functions $f_1(x) = x^{1/3}$, $f_2(x) = x^{2/3}$ examined in Sec. 87 we have (for $x \gtrless 0$)

$$f_1'(x) = \frac{1}{3x^{2/3}}, \quad f_2'(x) = \frac{2}{3x^{1/3}}.$$

Since the first expression tends to $+\infty$ as $x \to 0$ and the second has the limits $\pm \infty$ as $x \to \pm 0$, respectively, we infer at once that $f_1(x)$ has a two-sided derivative $+\infty$ at the point $x = 0$ while for $f_2(x)$ there exist at this point one-sided derivatives only: $+\infty$ from the right and $-\infty$ from the left.

It follows from the above statements that if a finite derivative $f'(x)$ exists in an interval, then it constitutes a function which cannot possess ordinary discontinuities or jumps; at all points it is either continuous or has a discontinuity of the second kind [Sec. 88, *(2)*].

104. Generalized theorem on finite increments. Cauchy generalized the theorem on finite increments stated in the preceding section in the following way.

CAUCHY'S THEOREM. *Suppose that (1) the functions $f(x)$ and $g(x)$ are defined and continuous in the closed interval $[a, b]$, (2) there exist finite derivatives $f'(x)$ and $g'(x)$ at least in the open interval (a, b), (3) $g'(x) \neq 0$ in the interval (a, b).*

Then between a and b a point c can be found, such that

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}. \tag{5}$$

This formula is called *Cauchy's formula.*

Proof. We first establish that the denominator of the left side of our relation is not zero, for otherwise this expression would have no meaning. If we had $g(b)=g(a)$, according to Rolle's theorem the derivative $g'(x)$ would vanish at an intermediate point, and this contradicts condition (3). Consequently $g(b)\neq g(a)$.

Consider now the auxiliary function

$$F(x)=f(x)-f(a)-\frac{f(b)-f(a)}{g(b)-g(a)}[g(x)-g(a)].$$

It satisfies all conditions of Rolle's theorem. In fact, $F(x)$ is continuous in $[a, b]$, for $f(x)$ and $g(x)$ are continuous; the derivative $F'(x)$ exists in (a, b), and it is equal to

$$F'(x)=f'(x)-\frac{f(b)-f(a)}{g(b)-g(a)}g'(x).$$

Finally, a direct substitution proves that $F(a)=F(b)=0$. Applying the above-mentioned theorem we infer the existence of a point c between a and b, at which $F'(c)=0$. In other words,

$$f'(c)-\frac{f(b)-f(a)}{g(b)-g(a)}g'(c)=0,$$

or

$$f'(c)=\frac{f(b)-f(a)}{g(b)-g(a)}g'(c).$$

Dividing by $g'(c)$ (this is admissible, since $g'(c)\neq 0$) we arrive at the required relation.

It is clear that Lagrange's theorem is a particular case of Cauchy's theorem. To deduce the formula on finite increments from Cauchy's formula it is sufficient to set $g(x)=x$.

In the theorems of Secs. 101, 102, 104 there appears under the sign of the derivative a mean value of the independent variable which, as was already indicated, is generally unknown. It provides also the derivative with a sort of mean value. For this reason the theorems are called "mean value theorems".

§ 2. Taylor's formula

105. Taylor's formula for a polynomial. If $p(x)$ is a polynomial of degree n,

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n; \tag{1}$$

differentiating it successively n times we have

$$p'(x) = a_1 + 2 \cdot a_2 x + 3 \cdot a_3 x^2 + \ldots + n \cdot a_n x^{n-1},$$

$$p''(x) = 1 \cdot 2 \cdot a_2 + 2 \cdot 3 \cdot a_3 x + \ldots + (n-1) n \cdot a_n x^{n-2},$$

$$p'''(x) = 1 \cdot 2 \cdot 3 \cdot a_3 + \ldots + (n-2) \cdot (n-1) n \cdot a_n x^{n-3},$$

. . . . . .

$$p^{(n)}(x) = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n \cdot a_n;$$

setting in all the above formulae $x = 0$ we find expressions for the coefficients of the polynomial in terms of the values of the polynomial and its derivatives at $x = 0$:

$$a_0 = p(0), \quad a_1 = \frac{p'(0)}{1!}, \quad a_2 = \frac{p''(0)}{2!},$$

$$a_3 = \frac{p'''(0)}{3!}, \ \ldots, \ a_n = \frac{p^{(n)}(0)}{n!}.$$

Let us substitute these values in (1):

$$p(x) = p(0) + \frac{p'(0)}{1!} x + \frac{p''(0)}{2!} x^2 + \frac{p'''(0)}{3!} x^3 + \ldots + \frac{p^{(n)}(0)}{n!} x^n. \tag{2}$$

This formula differs from (1) in the notation of the coefficients.

Instead of expanding the polynomial with respect to the powers of x we can take its expansion with respect to $x - x_0$ where x_0 is a constant particular value of x:

$$p(x) = A_0 + A_1 (x - x_0) + A_2 (x - x_0)^2 + A_3 (x - x_0)^3 + \ldots + A_n (x - x_0)^n. \tag{3}$$

Setting $x - x_0 = \xi$, $p(x) = p(x_0 + \xi) = P(\xi)$, we have for the coefficients of the polynomial

$$P(\xi) = A_0 + A_1\xi + A_2\xi^2 + A_3\xi^3 + \ldots + A_n\xi^n,$$

by statement proved above, the expressions

$$A_0 = P(0), \quad A_1 = \frac{P'(0)}{1!}, \quad A_2 = \frac{P''(0)}{2!},$$

$$A_3 = \frac{P'''(0)}{3!}, \ \ldots, \ A_n = \frac{P^{(n)}(0)}{n!}.$$

But

$$P(\xi) = p(x_0 + \xi), \quad P'(\xi) = p'(x_0 + \xi),$$

$$P''(\xi) = p''(x_0 + \xi), \ \ldots,$$

and hence

$$P(0) = p(x_0), \quad P'(0) = p'(x_0), \quad P''(0) = p''(x_0), \ \ldots$$

and

$$A_0 = p(x_0), \quad A_1 = \frac{p'(x_0)}{1!}, \quad A_2 = \frac{p''(x_0)}{2!},$$

$$A_3 = \frac{p'''(x_0)}{3!}, \ \ldots, \ A_n = \frac{p^{(n)}(x_0)}{n!}, \qquad (4)$$

i.e. the coefficients of expansion (3) are expressed by the values of the polynomial and its derivatives at $x = x_0$.

Substitute into (3) the expressions (4); then

$$p(x) = p(x_0) + \frac{p'(x_0)}{1!}(x - x_0) + \frac{p''(x_0)}{2!}(x - x_0)^2$$

$$+ \frac{p'''(x_0)}{3!}(x - x_0)^3 + \ldots + \frac{p^{(n)}(x_0)}{n!}(x - x_0)^n. \qquad (5)$$

Formula (5), similar to (2) for a particular case of $x_0 = 0$, is called *Taylor's formula*. Incidentally, formula (2) is usually called *Maclaurin's formula*†. Taylor's formula has important applications in algebra.

---

† Brook Taylor (1685–1731) and Colin Maclaurin (1698–1746)—British mathematicians, followers of Newton.

We now make the obvious remark (which will be useful in future considerations) that if the polynomial $p(x)$ is represented in the form

$$p(x) = c_0 + \frac{c_1}{1!}(x - x_0) + \frac{c_2}{2!}(x - x_0)^2 + \frac{c_3}{3!}(x - x_0)^3 + \ldots + \frac{c_n}{n!}(x - x_0)^n,$$

we necessarily have

$$p(x_0) = c_0, \quad p'(x_0) = c_1, \quad p''(x_0) = c_2, \quad \ldots, \quad p^{(n)}(x_0) = c_n.$$

106. Expansion of an arbitrary function. We now proceed to investigate an arbitrary function $f(x)$ which is not in general a polynomial but is defined over an interval $\mathfrak{X}$. Assume that there exist at the point x_0 (of $\mathfrak{X}$) its derivatives of all orders up to the nth, including the latter. More precisely this means that the function has the derivatives of all orders up to the $(n-1)$th, including the latter

$$f'(x), f''(x), f'''(x), \ldots, f^{(n-1)}(x),$$

in a vicinity of point x_0 and moreover it has the derivative of nth order $f^{(n)}(x_0)$ at the point x_0 itself†. Then, by virtue of (5) we can construct for $f(x)$ the polynomial

$$p_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n. \quad (6)$$

According to the remark made above, this polynomial and its derivatives (up to the nth inclusive) at the point x_0 have the same values as the function $f(x)$ and its derivatives.

Now, however, if the function $f(x)$ itself is not a polynomial of nth degree, we cannot say that $f(x) = p_n(x)$. The polynomial $p_n(x)$ yields only an approximation to the function $f(x)$, by means

† If x_0 is an end-point of the interval $\mathfrak{X}$, then speaking of the derivatives at this point we must consider one-sided derivatives; similarly, by the neighbourhood of point x_0 in this case we mean a one-sided neighbourhood.

of which it can be calculated with a certain degree of accuracy. In this connection it is of interest to estimate the difference

$$r_n(x) = f(x) - p_n(x)$$

or

$$r_n(x) = f(x) - f(x_0) - \frac{f'(x_0)}{1!}(x - x_0) - \ldots - \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \qquad (7)$$

for a given x from $\mathfrak{X}$ and a given n.

The above expression for $r_n(x)$ cannot serve this purpose. To represent it in a form more convenient for investigation, we shall have to impose upon the function $f(x)$ more restricting conditions than those which are directly required for the construction of the polynomial $p_n(x)$ itself. Namely, we shall assume henceforth that there exist for the function $f(x)$ in $\mathfrak{X}$ all derivatives up to $(n+1)$th

$$f'(x),\ f''(x),\ \ldots,\ f^{(n)}(x),\ f^{(n+1)}(x).$$

Let us now fix an arbitrary value of x from the interval $\mathfrak{X}$ and, in the right-hand side of formula (7), replacing the constant number x_0 by the variable z we construct a new, auxiliary function

$$\varphi(z) = f(x) - f(z) - \frac{f'(z)}{1!}(x - z) - \frac{f''(z)}{2!}(x - z)^2 - \ldots - \frac{f^{(n)}(z)}{n!}(x - z)^n,$$

regarding the independent variable z as varying over the interval $[x_0, x]$†. In this interval the function $\varphi(z)$ is continuous and takes at its ends the values [see (7)]

$$\varphi(x_0) = r_n(x), \quad \varphi(x) = 0. \qquad (8)$$

Furthermore, in the interval (x_0, x) there exists the derivative

$$\varphi'(z) = -f'(z) - \left[\frac{f''(z)}{1!}(x - z) - f'(z)\right] - \left[\frac{f'''(z)}{2!}(x - z)^2 - \frac{f''(z)}{1!}(x - z)\right] - \ldots - \left[\frac{f^{(n+1)}(z)}{n!}(x - z)^n - \frac{f^{(n)}(z)}{(n-1)!}(x - z)^{n-1}\right]$$

† For definiteness we assume that $x > x_0$.

or, after simplification,

$$\varphi'(z) = -\frac{f^{(n+1)}(z)}{n!}(x-z)^n.$$

If we now take a new function $\psi(z)$ which is continuous in the interval $[x_0, x]$ and has in the open interval (x_0, x) a non-vanishing derivative, we may apply to the pair of functions $\varphi(z), \psi(z)$ the Cauchy formula [Sec. 104]

$$\frac{\varphi(x_0)-\varphi(x)}{\psi(x_0)-\psi(x)} = \frac{\varphi'(c)}{\psi'(c)},$$

where c lies between x_0 and x, i.e. $c = x_0 + \theta(x-x_0)$ $(0<\theta<1)$. From (8) and (9) we find that

$$r_n(x) = -\frac{\psi(x_0)-\psi(x)}{\psi'(c)}\,\frac{f^{(n+1)}(c)}{n!}(x-c)^n. \tag{10}$$

Select the function $\psi(z)$ in such a way that

$$\psi(z) = (x-z)^{n+1};$$

then the conditions stated for $\psi(z)$ are satisfied. We have

$$\psi(x_0) = (x-x_0)^{n+1}, \quad \psi(x) = 0, \quad \psi'(c) = -(n+1)(x-c)^n.$$

Substituting into (10) we finally obtain

$$r_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}. \tag{11}$$

Now, taking into account (7) and (11) we can represent the function $f(x)$ by the formula

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}, \tag{12}$$

which differs from Taylor's formula for a polynomial by the presence of *the remainder term* (11).

The form (11) of the remainder term is due to Lagrange; the remainder term in this formula resembles the next successive term of Taylor's formula, only instead of computing the $(n+1)$th

derivative at point x_0 it is taken for a mean value of c, between x_0 and x.

Formula (12) is called Taylor's formula with the additional term in *Lagrange's form.* If we take $f(x_0)$ to the left-hand side and set $x - x_0 = \Delta x$, it takes the form

$$\Delta f(x_0) = f(x_0 + \Delta x) - f(x_0) = \frac{f'(x_0)}{1!}\Delta x + \frac{f''(x_0)}{2!}\Delta x^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}\Delta x^n + \frac{f^{(n+1)}(c)}{(n+1)!}\Delta x^{n+1}. \quad (12a)$$

In this form it is a direct generalization of the formula on finite increments [Sec. 102, (4)],

$$\Delta f(x_0) = f(x_0 + \Delta x) - f(x_0) = f'(c)\,\Delta x,$$

which corresponds to $n = 0$.

Although the remainder term in Lagrange's form is very simple indeed, in certain cases this form is not suitable for estimating the remainder and it is necessary to use less simple forms. We mention here the remainder term in the *Cauchy form.* It is derived from (10) by setting $\psi(z) = x - z$. Then

$$\psi(x_0) = x - x_0, \quad \psi(x) = 0, \quad \psi'(c) = -1$$

and since

$$(x - c)^n = [x - x_0 - \theta(x - x_0)]^n = (1 - \theta)^n\,(x - x_0)^n,$$

we arrive at the final expression

$$r_n(x) = \frac{f^{(n+1)}[x_0 + \theta(x - x_0)]}{n!}\,(1 - \theta)^n (x - x_0)^{n+1}. \quad (13)$$

Notwithstanding the loss (as compared with Lagrange's form) of the factor $n + 1$ in the denominator, this form sometimes is more convenient, owing to the presence of the factor $(1 - \theta)^n$.

Taylor's formula with the remainder term in some form is a sort of mean value formula; it contains the means c and θ.

107. Another form for the remainder term. The forms of the remainder term in Taylor's formula introduced in the preceding section are used when for some fixed values of x (distinct from x_0) we want to replace a function $f(x)$ by a polynomial $p_n(x)$ and numerically estimate the resulting error. It may happen however that we

are not interested in definite values of x, but it is important to possess a definite knowledge of the behaviour of the remainder term when x tends to x_0, or more precisely its order of smallness is important. This order can be established under even somewhat weaker conditions than above. Namely we assume that only n successive derivatives

$$f'(x), f''(x), \ldots, f^{(n)}(x)$$

exist in the neighbourhood (two-sided or one-sided) of the point x_0 and the last derivative is continuous at x_0†. Then, replacing n by $n-1$ in formula (12) we have

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \ldots + \frac{f^{(n-1)}(x_0)}{(n-1)!}(x-x_0)^{n-1} + \frac{f^{(n)}(c)}{n!}(x-x_0)^n,$$

where c lies between x_0 and x. Set in the last term

$$\frac{f^{(n)}(c)}{n!} = \frac{f^{(n)}(x_0)}{n!} + \alpha(x); \tag{14}$$

since it is evident that $c \to x_0$ as $x \to x_0$, and hence (by continuity $f^{(n)}(c) \to f^{(n)}(x_0)$. Moreover, $\alpha(x) \to 0$ and $\alpha(x)(x-x_0)^n = o[(x-x_0)^n]$. Thus we finally obtain

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + o[(x-x_0)^n]. \tag{15}$$

Consequently, now

$$r_n(x) = o[(x-x_0)^n], \tag{16}$$

i.e. we know that for a constant n the remainder term is an infinitesimal of order higher than n, as $x \to x_0$, although, which is important, we do not know anything about its magnitude for any

† In fact, it is sufficient to assume only the existence of the derivative $f^{(n)}(x_0)$ at one point $x = x_0$. We have imposed stronger conditions to simplify the reasoning.

fixed value of x. The form (16) of the remainder term was given by Peano†.

We observe that formula (15) has, in fact, a definitely "local" nature and describes only the behaviour of the function as x tends to x_0.

If in (15) we again take $f(x_0)$ to the left side and we set $x - x_0 = \Delta x$, we arrive at the expansion

$$\Delta f(x_0) = f'(x_0)\Delta x + \frac{f''(x_0)}{2!}\Delta x^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}\Delta x^n + o(\Delta x^n), \quad (15a)$$

which is a generalization of formula (3) of Sec. 82,

$$\Delta f(x_0) = f'(x_0)\Delta x + o(\Delta x),$$

which follows if we set $n = 1$.

Sometimes it is convenient to take instead of (14)

$$f_n(c) = f_n(x_0) + \alpha(x);$$

here again $\alpha(x) \to 0$ as $x \to x_0$ and Taylor's formula, with the remainder term in the Peano form, takes the form

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \ldots + \frac{f^{(n-1)}(x_0)}{(n-1)!}(x - x_0)^{n-1} + \frac{f^{(n)}(x_0) + \alpha(x)}{n!}(x - x_0)^n. \quad (17)$$

We make one final remark. If we replace Δx by dx in formulae (12a) and (15a) and remember that

$$f'(x_0)dx = df(x_0), \quad f''(x_0)dx^2 = d^2f(x_0), \quad \ldots, \quad f^{(n)}(x_0)dx^n = d^nf(x_0)$$

and

$$f^{(n+1)}(c)dx^{n+1} = d^{n+1}f(c),$$

then, substituting, we represent the expansion in the form

$$\Delta f(x_0) = df(x_0) + \frac{1}{2!}d^2f(x_0) + \ldots + \frac{1}{n!}d^nf(x_0) + \frac{1}{(n+1)!}d^{n+1}f(c) \quad (c = x_0 + \theta\Delta x, \quad 0 < \theta < 1) \quad (12b)$$

or

$$\Delta f(x_0) = df(x_0) + \frac{1}{2!}d^2f(x_0) + \ldots + \frac{1}{n!}d^nf(x_0) + o(\Delta x^n). \quad (15b)$$

† Giuseppe Peano (1858–1932)—an Italian mathematician.

Thus, if we assume that $\Delta x \to 0$, according to these formulae, from the infinitesimal increment of the function $\Delta f(x_0)$, not only its principal term—the first differential—is separated out, but also the terms of higher orders of smallness which, to within the factorials in the denominators, are identical with the successive higher differentials

$$d^2 f(x_0), \ \ldots, \ d^n f(x_0).$$

108. Application of the derived formulae to elementary functions. The simplest form of Taylor's formula occurs when $x_0 = 0$†:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + r_n(x). \qquad (18)$$

We can always reduce the problem to this particular case, taking $x - x_0$ as the new independent variable.

Let us now examine some actual expansions of elementary functions, using the above formulae.

(1) Suppose that $f(x) = e^x$; then $f^{(k)}(x) = e^x$ for $k = 1, 2, 3, \ldots$ [Sec. 96, *(3)*]. Since in this case $f(0) = 1$, $f^{(k)}(0) = 1$, by (18)

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + r_n(x).$$

(2) If $f(x) = \sin x$, then $f^{(k)}(x) = \sin(x + k\pi/2)$ [Sec. 96, *(4)*]; hence

$$f(0) = 0, \quad f^{(2m)}(0) = \sin m\pi = 0,$$

$$f^{(2m-1)}(0) = \sin\left(m\pi - \frac{\pi}{2}\right) = (-1)^{m-1} \qquad (m = 1, 2, 3, \ldots).$$

Therefore, setting in formula (18) $n = 2m$ we have

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^{m-1}\frac{x^{2m-1}}{(2m-1)!} + r_{2m}(x).$$

(3) Similarly, when $f(x) = \cos x$ [Sec. 96, *(4)*]

$$f^{(k)}(x) = \cos\left(x + k \cdot \frac{\pi}{2}\right); \quad f(0) = 1, \ f^{(2m)}(0) = (-1)^m,$$

$$f^{(2m-1)}(0) = 0 \qquad (m = 1, 2, 3, \ldots).$$

† This formula is also attributed to Maclaurin.

Thus (taking $n = 2m + 1$)

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^m \frac{x^{2m}}{2m!} + r_{2m+2}.$$

(4) Now consider the power function x^m where m is not a positive integer or zero. Now as $x \to 0$, either the function itself (if $m < 0$) or its derivatives (beginning from a certain order n, where $n > m$) increase infinitely. Consequently we cannot take $x_0 = 0$.

Set $x_0 = 1$, i.e. we expand x^m with respect to the powers of $x - 1$. It was incidentally mentioned before that we may introduce $x - 1$ as the new independent variable; we shall denote it as before by x, and we expand the function $(1 + x)^m$ with respect to the powers of x.

We know that [Sec. 96, *(1)*]

$$f^k(x) = m(m-1)\ldots(m-k+1)(1+x)^{m-k}.$$

Hence

$$f(0) = 1, \quad f^{(k)}(0) = m(m-1)\ldots(m-k+1).$$

The expansion has the form

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{1 \cdot 2} x^2 + \ldots + \frac{m(m-1)\ldots(m-n+1)}{1 \cdot 2 \ldots n} x^n + r_n(x).$$

(5) Consider now the logarithmic function $\log x$ which tends to $-\infty$ as $x \to +0$; as before, we shall examine the function $f(x) = \log(1 + x)$ and we expand it with respect to the powers of x.

Thus [Sec. 96, *(2)*]

$$f^{(k)}(x) = \frac{(-1)^{k-1}(k-1)!}{(1+x)^k} \dagger,$$

$$f(0) = 0, \quad f^{(k)}(0) = (-1)^{k-1}(k-1)!.$$

Consequently,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-1} \frac{x^n}{n} + r_n(x).$$

† We imply that $0! = 1$ always.

(6) Suppose now that $f(x) = \arctan x$. It is easy to derive from Sec. 96, *(5)* the values of its derivatives for $x = 0$:

$$f^{(2m)}(0) = 0, \quad f^{(2m-1)}(0) = (-1)^{m-1}(2m-2)!$$

and the expansion takes the form

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^{m-1}\frac{x^{2m-1}}{2m-1} + r_{2m}(x).$$

109. Approximate formulae. Examples. If we disregard the remainder term in formula (18) we are led to the approximate formula

$$f(x) \doteq f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n,$$

replacing a general function by a polynomial. The accuracy of this approximation can be estimated in two ways. Either the magnitude of the bound of the error $r_n(x)$ is determined making use of the Lagrangian form of the remainder term,

$$r_n(x) = \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1} \qquad (0 < \theta < 1),$$

or, following Peano, one finds the order of smallness of this error as $x \to 0$,

$$r_n(x) = o(x^n).$$

As examples we consider the above expansion of the elementary functions.

(1) Consider $f(x) = e^x$. The approximate formula is

$$e^x \doteq 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!};$$

since the remainder term has the form

$$r_n(x) = \frac{e^{\theta x}}{(n+1)!}x^{n+1},$$

then, for instance, when $x > 0$, the estimate of the error is

$$0 < r_n(x) < e^x \frac{x^{n+1}}{(n+1)!}.$$

In particular, if $x = 1$

$$e \doteq 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}, \quad 0 < r_n(1) < \frac{3}{(n+1)!}.$$

A similar formula was already employed in Sec. 49 for an approximate calculation of the number e, but the estimate of the remainder term derived in another way was there more exact.

(2) Taking $f(x) = \sin x$ we obtain

$$\sin x \doteq x - \frac{x^3}{3!} - \frac{x^5}{5!} - \ldots + (-1)^{m-1}\frac{x^{2m-1}}{(2m-1)!}.$$

In this case the remainder term is

$$r_{2m}(x)=\frac{\sin\left(\theta x+(2m+1)\frac{\pi}{2}\right)}{(2m+1)!}x^{2m+1}=(-1)^m\cos\theta x\frac{x^{2m+1}}{(2m+1)!}$$

and the error is easily estimated, namely

$$|r_{2m}(x)|\leqslant\frac{|x|^{2m+1}}{(2m+1)!}.$$

In particular, if we take one term only,

$$\sin x\doteq x;$$

in order that the error be smaller than, say, 0.001, it is sufficient to take (for $x>0$)

$$\frac{x^3}{6}<0.001 \quad \text{or} \quad x<0.1817,$$

which approximately is equal to 10°. Making use of the two-term formula

$$\sin x\doteq x-\frac{x^3}{6},$$

to obtain the same accuracy it is sufficient to take

$$\frac{x^5}{120}<0.001 \quad \text{or} \quad x<0.6544 \quad (\doteq 37.5°);$$

if we confine ourselves to the angles $x<0.4129$ ($\doteq 23.5°$) the error is even smaller than 0.0001, etc.

(3) Similarly, for $f(x)=\cos x$ we have

$$\cos x\doteq 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots+(-1)^m\frac{x^{2m}}{2m!},$$

and

$$r_{2m+1}(x)=(-1)^{m+1}\cos\theta x\frac{x^{2m+2}}{(2m+2)!}.$$

Hence

$$|r_{2m+1}(x)|\leqslant\frac{|x|^{2m+2}}{(2m+2)!}.$$

For instance, for the formula

$$\cos x\doteq 1-\frac{x^2}{2}$$

we have the error

$$|r_3(x)|\leqslant\frac{x^4}{24}$$

and it will certainly be, say, <0.0001 for $x<0.2213$ ($\doteq 13°$), and so on.

We draw the reader's attention to the essential progress achieved, as compared with the formulae of Secs. 56, 57, 93; now we can find the bounds of the error and we possess formulae of arbitrary accuracy.

Finally, we give an example of an approximate formula of an entirely different type which, however, makes use of Taylor's formula.

(*4*) To rectify approximately an arc of a circle which is small compared with the radius (Fig. 41) Tchebychev† constructed the following rule: the arc s is approximately equal to the sum of the equal sides of the isosceles triangle constructed on the chord d, the height of which is $\sqrt{(4/3)\cdot f}$.

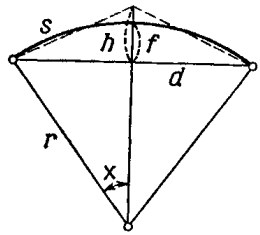

FIG. 41.

Denoting half of the central angle by x and the radius of the arc by r, we have $s = 2rx$. On the other hand,

$$\frac{1}{2}d = r\sin x = r\left\{x - \frac{x^3}{6} + o(x^4)\right\}, \quad \left(\frac{1}{2}d\right)^2 = r^2\left\{x^2 - \frac{x^4}{3} + o(x^5)\right\},$$

$$h = \sqrt{\left(\frac{4}{3}\right)\cdot f} = \sqrt{\left(\frac{4}{3}\right)\cdot r(1-\cos x)} = \sqrt{\left(\frac{4}{3}\right)\cdot r\left\{\frac{1}{2}x^2 + o(x^3)\right\}},$$

$$h^2 = r^2\left\{\frac{x^4}{3} + o(x^5)\right\},$$

and therefore the above-mentioned sum of the sides, by the theorem of Pythagoras, is equal to

$$2\sqrt{\left\{\left(\frac{1}{2}d\right)^2 + h^2\right\}} = 2r\sqrt{[x^2 + o(x^5)]} = 2rx\sqrt{[1 + o(x^3)]} = 2rx + o(x^4).$$

It is clear that the purpose of the factor in the Tchebychev formula lies in the fact that under the root sign the term in x^4 has dropped out. Finally, the approximate value of the arc that we have derived differs from the arc itself by a quantity of the fourth order of smallness.

We shall return to Taylor's formula with the remainder term in Chapter 15 (second volume) on infinite series, where this formula will play an important role. We shall there also give examples of applications of series to approximate computations which are frequently, in fact, merely applications of the Taylor formula.

† Academician Pafnutii Lvovitch Tchebychev (1821–1894)—a great Russian mathematician, the originator of the St. Petersburg mathematical school.

CHAPTER 7

INVESTIGATION OF FUNCTIONS BY MEANS OF DERIVATIVES

§ 1. Investigation of the behaviour of functions

110. Conditions that a function may be constant. In investigating the behaviour of functions, there first arises the problem of determination of the conditions according to which a function has a constant value, or varies monotonically [Sec. 47] in a prescribed interval.

THEOREM. *Suppose that the function $f(x)$ is defined in the interval $\mathfrak{X}$† and has inside it a finite derivative $f'(x)$ and is continuous at the end-points (if they belong to $\mathfrak{X}$). In order that $f(x)$ (in $\mathfrak{X}$) be a constant, it is sufficient that*

$$f'(x) = 0 \text{ inside } \mathfrak{X}.$$

Proof. Suppose that the condition is satisfied. Fix a point x_0 in $\mathfrak{X}$ and take an arbitrary point $x \neq x_0$. In the interval $[x_0, x]$ or $[x, x_0]$ all conditions of Lagrange's theorem are satisfied [Sec. 102]. Consequently we may write

$$f(x) - f(x_0) = f'(c)(x - x_0),$$

c being between x_0 and x, and hence it certainly lies inside $\mathfrak{X}$. But, according to the assumption $f'(c) = 0$ and therefore for all x from $\mathfrak{X}$ we have

$$f(x) = f(x_0) = \text{const},$$

which proves our proposition.

Observe that it is evident that the above condition is also necessary for a function to be constant.

The following simple corollary has an important application in the integral calculus.

† It can be open or otherwise, finite or infinite.

COROLLARY. *Suppose that two functions $f(x)$ and $g(x)$ are defined in the interval $\mathfrak{X}$, inside it have finite derivatives $f'(x)$ and $g'(x)$, and the functions are continuous at the end-points (if they belong to $\mathfrak{X}$). If, moreover,*

$$f'(x) = g'(x) \text{ inside } \mathfrak{X},$$

then the functions differ by only a constant over the whole interval $\mathfrak{X}$, i.e.

$$f(x) = g(x) + C \quad (C = \text{const}).$$

To prove this corollary it suffices to apply the theorem to the difference $f(x) - g(x)$; since its derivative $f'(x) - g'(x)$ vanishes inside $\mathfrak{X}$, the difference itself is a constant.

Consider, as an example, the functions

$$\arctan x \quad \text{and} \quad \frac{1}{2}\arctan\frac{2x}{1-x^2}.$$

It can easily be verified that their derivatives are identical at all points x, except $x = \pm 1$ (where the second function has no meaning). Consequently the identity

$$\frac{1}{2}\arctan\frac{2x}{1-x^2} = \arctan x + C$$

holds only for each of the intervals

$$\mathfrak{X}_1 = (-1, 1), \quad \mathfrak{X}_2 = (-\infty, -1), \quad \mathfrak{X}_3 = (1, +\infty)$$

separately. It is interesting also that the values of the constant C for these intervals are distinct. For the first $C = 0$ (which is verified by setting $x = 0$) while for the others $C = \pi/2$ or $C = -\pi/2$ (which can easily be verified by passing to the limits, $x \to -\infty$ or $+\infty$).

All these relations can also be proved in an elementary way.

Remark. The value of this theorem is revealed in theoretical investigations and generally in the cases when the function is prescribed in such a way that it does not directly follow from its definition that it has a constant value. Such cases will often occur in subsequent considerations.

111. Condition of monotonicity of a function. We shall now determine how we can establish from its derivative whether a function is increasing (or decreasing) in a given interval.

THEOREM. *Suppose that the function $f(x)$ is defined in an interval $\mathfrak{X}$, has inside it a finite derivative $f'(x)$, and that it is continuous at its end-points (if they belong to $\mathfrak{X}$). In order that $f(x)$ is monotonically*

increasing (decreasing), in the narrow sense, in $\mathfrak{X}$ *it is sufficient that*

$$f'(x) > 0 \ (< 0) \ \textit{inside} \ \mathfrak{X}.$$

Proof. This will be carried out for the case of an increasing function. Suppose that the indicated condition is satisfied. Take two values x' and x'' $(x' < x'')$ from $\mathfrak{X}$ and apply Lagrange's formula to the function $f(x)$ over the interval $[x', x'']$:

$$f(x'') - f(x') = f'(c)(x'' - x') \qquad (x' < c < x'').$$

Since $f'(c) > 0$ we have

$$f(x'') > f(x'),$$

and the function $f(x)$ is strictly increasing.

Now the stated condition is not entirely necessary. The statement of the theorem remains valid also for instance in the case when the derivative $f'(x)$ vanishes at a finite number of points inside the interval $\mathfrak{X}$. This is readily verified by applying the theorem separately to each part into which the basic interval is divided by these points.

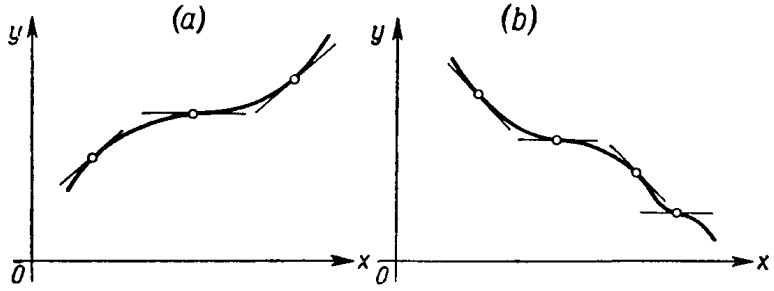

FIG. 42.

The established relation between the sign of the derivative and the direction of variation of the function is geometrically obvious, if we bear in mind [Secs. 77, 78] that the derivative represents the slope of the tangent to the graph of the function. The sign of the slope indicates whether the tangent is inclined upwards or downwards, and therefore whether the curve itself goes upwards or downwards (Fig. 42). At separate points the tangent may turn out to be horizontal; this corresponds to the vanishing of the derivative.

Examples. (1) A simple example of the fact just mentioned is provided by the function $f(x) = x^3$; it is increasing, but nevertheless its derivative $f'(x) = 3x^2$ vanishes at $x = 0$.

(2) Similarly, the function

$$f(x) = x - \sin x$$

is increasing, since its derivative

$$f'(x) = 1 - \cos x$$

is non-negative and vanishes for the values $x = 2k\pi$ $(k = 0, \pm 1, \pm 2, \ldots)$.

112. Maxima and minima; necessary conditions. If the function $f(x)$ defined and continuous in the interval $[a, b]$ is not monotonic, such parts $[\alpha, \beta]$ of the interval $[a, b]$ may be found, in which the greatest or the smallest value is attained by the function at an interior point, i.e. between α and β. On the graph of the function (Fig. 43) there correspond to such intervals characteristic crests and valleys.

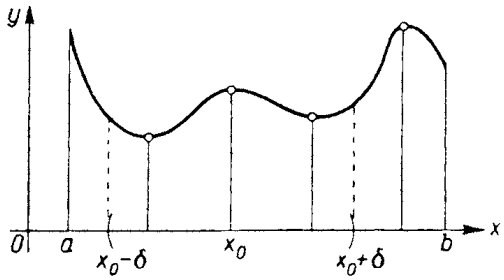

FIG. 43.

We say that a function $f(x)$ has at a point x_0 a maximum (or a minimum) if this point can be surrounded by a neighbourhood $(x_0 - \delta, x_0 + \delta)$ contained in the interval in which the considered function is defined, such that for all points x of this vicinity

$$f(x) \leqslant f(x_0) \quad (\text{or } f(x) \geqslant f(x_0)).$$

In other words, the point x_0 is a maximum (minimum) of the function $f(x)$ if the value $f(x_0)$ is the greatest (the smallest) of all values taken by the function in some neighbourhood of the point. Observe that the definition of the maximum (minimum) assumes that the function is given on both sides of the point x_0.

If there exists a neighbourhood inside the bounds of which (for $x \neq x_0$) the strict inequality

$$f(x) < f(x_0) \quad (\text{or } f(x) > f(x_0)),$$

holds, then we say that the function has at the point x_0 a *proper* maximum (minimum), while otherwise it has an *improper* one.

If the function has maxima at the points x_0 and x_1, then, applying to the interval $[x_0, x_1]$ the second Weierstrass theorem [Sec. 73], we find that the smallest value is taken by the function in the considered interval at a point x_2 between x_0 and x_1 and the minimum occurs at this point. Similarly, between two minima there is necessarily a maximum. In this (the simplest, and in applications the most important) case, when the function has in all only a finite number of maxima and minima, they simply alternate. To denote a maximum or a minimum we use the single term—an *extremum.*

Consider the problem of finding the values of the argument for which the function takes an extremum. In solving this problem an essential part is played by the concept of the derivative.

Assume first that the function $f(x)$ has in the interval (a, b) a finite derivative. If at a point x_0 the function has an extremum, then applying Fermat's theorem [Sec. 100] to the interval $(x_0 - \delta, x_0 + \delta)$, which we introduced above, we find that $f'(x_0) = 0$, which is the necessary condition for an extremum. The extremum should be sought only at the points at which the derivative vanishes; such points will be called *stationary*†.

However, one should not think that every stationary point provides the function with an extremum; the above necessary condition is not sufficient. For instance, we found in Sec. 111, *(1)* that the derivative $3x^2$ of the function x^3 vanishes for $x = 0$ but at this point the function has no extremum; it increases all the time.

Thus we can say only that the stationary point of a function $f(x)$ is "suspect" and should undergo a further investigation.

If we extend the class of considered functions to admit those which have no finite derivative at isolated points, it is possibile that the extremum may occur at one of these points. They should

† At these points the variation of the function as it "stops"; the velocity of the variation is zero [Sec. 78].

also therefore be classified among "the suspicious points" and should be investigated.

113. The first rule. Suppose that we suspect that the point x_0 may be an extremum of the function $f(x)$.

Assume that in a neighbourhood $(x_0 - \delta, x_0 + \delta)$ of this point (at least for all $x \neq x_0$) there exists a finite derivative $f'(x)$, and both on the left of x_0 and on the right of this point (separately) it has a constant sign. Then the following three cases are possible.

I. $f'(x) > 0$ for $x < x_0$ and $f'(x) < 0$ for $x > x_0$, i.e. derivative $f'(x)$ changes its sign from plus to minus in passing through the point x_0. In this case in the interval $[x_0 - \delta, x_0]$ the function $f(x)$ increases, while in the interval $[x_0, x_0 + \delta]$ it decreases; hence the greatest value of $f(x)$ occurs in the interval $[x_0 - \delta, x_0 + \delta]$, i.e. at the point x_0 the function $f(x)$ has a maximum.

II. $f'(x) < 0$ for $x < x_0$ and $f'(x) > 0$ for $x > x_0$, i.e. the derivative $f'(x)$ changes its sign from minus to plus in passing through the point x_0. In an analogous way we find in this case that at the point x_0 the function has a minimum.

III. $f'(x) > 0$ both for $x < x_0$ and $x > x_0$, or $f'(x) < 0$ both on the left and on the right of x_0, i.e. $f'(x)$ does not change its sign in passing through point x_0. Then the function either increases all the time or all the time it decreases; in an arbitrary neighbourhood of the point x_0, on one side points x can be found at which $f(x) < f(x_0)$, while on the other side points x at which $f(x) > f(x_0)$; thus there is no extremum at the point x_0.

Consequently we have the first rule for the investigation of the "suspect" values of x_0: substituting into the derivative $f'(x)$ first $x < x_0$ and then $x > x_0$ we establish the sign of the derivative near the point x_0 on the left and on the right of it; if the derivative $f'(x)$ changes its sign from plus to minus, then a maximum occurs; if on the other hand the sign changes from minus to plus, a minimum occurs. If, finally, no change of sign occurs there is no extremum.

We shall now describe the class of functions to which the above rule will be applied. The function $f(x)$ is assumed to be continuous in the interval $[a, b]$ and having there a continuous derivative $f'(x)$, except possibly at a finite number of points. At these points the derivative $f'(x)$ tends to infinite limits, both from the right and

from the left, the limits having the same or different signs; in the first case there exists a two-sided infinite derivative, while, in the latter case, there exist one-sided derivatives with distinct signs†. Finally, we also assume that the derivative vanishes at a finite number of points. The graphic illustration of the various possibilities for the "suspect" points is given in Fig. 44.

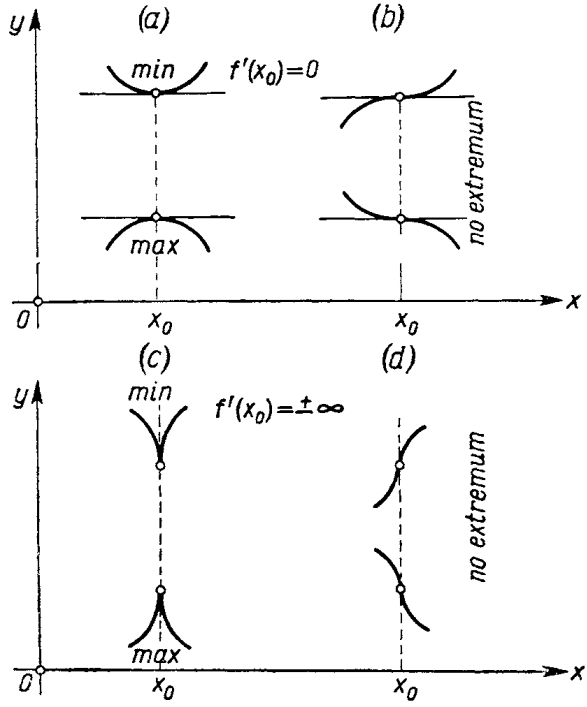

FIG. 44.

We see that in the cases (*b*), (*c*), (*d*) the curve intersects the tangent, passing from one to the other side of it; in these cases it is said that the function has a *point of inflection.*

For functions of the above class the rule just given completely solves the stated problem. The essential fact is that for such a function

† We can distinguish between these two cases by considering the sign of the derivative—as just mentioned.

in the interval (a, b) there are only a finite number of stationary points or points at which the finite derivative does not exist

$$a < x_1 < x_2 < \ldots < x_k < x_{k+1} < \ldots < x_{n-1} < b, \tag{1}$$

and in any interval

$$(a, x_1),\ (x_1, x_2),\ \ldots,\ (x_k, x_{k+1}),\ \ldots,\ (x_{n-1}, b) \tag{2}$$

the derivative $f'(x)$ has a constant sign. In fact, if $f'(x)$ changed its sign, for instance in the interval (x_k, x_{k+1}), then in view of the continuity of $f'(x)$ it would vanish (by the Bolzano–Cauchy theorem [Sec. 68]) at a point between x_k and x_{k+1}, which is impossible, since *all* roots of the derivative are contained in the sequence of points (1).

By virtue of the theorem of Sec. 111, the function varies strictly monotonically in every interval (2).

Remark. Although the above class of functions contains all the practically interesting cases, it is useful to realize that cases may be encountered when our rule of investigating the "suspect" points cannot be applied. If, for instance, we consider the function defined by the relations

$$f(x) = x^2 \sin\frac{1}{x} \quad \text{for} \quad x \neq 0 \quad \text{and} \quad f(0) = 0,$$

then we know [Sec. 88, *(2)*] that for $x = 0$ the function has a derivative $f'(0) = 0$. However, in an arbitrary neighbourhood of this stationary point, both on the left and on the right, the derivative

$$f'(x) = 2x \sin\frac{1}{x} - \cos\frac{1}{x}$$

vanishes an infinite number of times, and is thus of alternating sign. The rule therefore cannot be applied (although even without it it is clear that there is no extremum).

114. The second rule. If x_0 is a stationary point

$$f'(x_0) = 0$$

and the function $f(x)$ possesses not only the first derivative $f'(x)$ in the neighbourhood of this point, but also the second derivative $f''(x_0)$ at the point x_0 itself, then the whole investigation can be reduced to an investigation of the sign of the latter derivative, assuming that it does not vanish.

In fact, according to the definition of the derivative, taking into account (3) we have

$$f''(x_0) = \lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{f'(x)}{x - x_0}.$$

But by the result *(2)* of Sec. 37 the function

$$\frac{f'(x)}{x - x_0} \tag{4}$$

acquires the sign of its limit $f''(x_0)$ only when x (which is different from x_0) is sufficiently near $x_0 (|x - x_0| < \delta)$.

Suppose now that, say, $f''(x_0) > 0$; then the fraction (4) is positive for all considered values of x. But for $x < x_0$ the denominator $x - x_0 < 0$ and consequently the numerator $f'(x)$ is necessarily also negative; conversely, for $x > x_0$ we have $x - x_0 > 0$ and hence $f'(x) > 0$. In other words, we have found that the derivative $f'(x)$ changes its sign from minus to plus and hence, by the first rule, there is a minimum at the point x_0. Similarly we establish that if $f''(x) < 0$, there is a maximum at x_0.

Thus, we are in a position to state the second rule for the investigation of a "suspect" value of x_0 (we substitute x_0 into the second derivative $f''(x)$): if $f''(x_0) > 0$ the function has a minimum, if $f''(x_0) < 0$ the function has a maximum.

In general we cannot always apply this rule; for instance, it is certainly not applicable to points at which a finite first derivative does not exist (for then there is also no second derivative at these points). In the cases when the second derivative vanishes the rule is also useless. The solution of the problem then depends on the behaviour of the higher derivatives [Sec. 117].

115. Construction of the graph of a function. By determining the values of x at which the function $y = f(x)$ has extremum values we may construct the graph of a function which shows exactly the behaviour of the function for increasing x in the interval $[a, b]$.

Previously [Sec. 19] we constructed the graph by considering points taken more or less densely, but at random, and without taking into account the singularities of the graph (unknown beforehand). We are now in a position to establish with the help of the above

methods a number of "basic" points *peculiar to the examined graph.* We have in mind here first of all the turning points of the graph, i.e. the peaks of its crests and valleys, corresponding to the extremum values of the function. Incidentally, we should consider all points at which the tangent is either horizontal or vertical, even if they do not correspond to the extrema of the function.

We shall confine ourselves to functions $y = f(x)$ belonging to the class indicated in Sec. 113. Then the following operations should be carried out in order to construct the graph of such a function $y = f(x)$.

(1) Determination of the values of x for which the derivative $y' = f'(x)$ vanishes or is infinite (or at least there exist infinite one-sided derivatives), and investigation for the extrema.

(2) Calculation of the values of the function $y = f(x)$ itself, for the above values of x and for the end-points a and b of the considered interval.

It is convenient to compile the results in a table (see the examples below) with a necessary indication of the nature of the point: viz., maximum, minimum, $y' = 0$, $y' = +\infty$, $y' = -\infty$ and finally $y' = \pm\infty$ or $y' = \mp\infty$ (this is the conventional notation for the case when the infinite one-sided derivatives at a point are of opposite signs). One may also complete the above points of the graph by some other points, for instance the intersections of the graph with the axes.

After introducing on the graph all the above points (the number of which is usually small) we draw through them the curve taking into account all existing singularities. It should be borne in mind that in the intervals between them [see Sec. 113] the derivative has a constant sign and the graph increases or decreases everywhere.

The computations and the drawing of the curve are simplified if the function does not change its value when the sign of x is altered (an *even* function) and hence the graph *is symmetric with respect to the vertical axis.* A similar role may be played by a function *symmetric about the origin* which is expressed analytically by $f(x) = -f(-x)$ (when $f(x)$ is called an *odd* function).

The graph constructed in such a way does not describe the ordinates exactly but gives a general indication of the behaviour of the

function (which is our objective now) indicating exactly the intervals in which it is increasing and decreasing and also the points at which the rate of change of the function is zero or infinite.

116. Examples. *(1)* Find the extrema of the function

$$f(x) = \sin^3 x + \cos^3 x$$

and construct its graph.

Since the function has period 2π it suffices to consider the interval $[0, 2\pi]$ of x. The derivative is

$$f'(x) = 3\ \sin^2 x \cdot \cos x - 3\ \sin x \cdot \cos^2 x = 3\ \sin x \cdot \cos x \cdot (\sin x - \cos x).$$

The zeros of the derivative (the stationary points) are the following:

$$0,\ \frac{\pi}{4},\ \frac{\pi}{2},\ \pi,\ \frac{5\pi}{4},\ \frac{3\pi}{2},\ (2\pi).$$

In passing through $x = 0$ the factor $\sin x$ changes its sign from minus to plus, and the whole derivative changes its sign from plus to minus, since the last two factors are negative near $x = 0$; thus there is a *maximum* at $x = 0$. The factor $\sin x - \cos x$, which vanishes for $x = \pi/4$, changes sign from minus to plus in passing through this point. The same holds for the derivative, since the first two factors are positive; consequently there is a *minimum* at $x = \pi/4$. Similarly we investigate the remaining stationary points; they are in turn points of maxima and minima of the function.

Instead of investigating the changes of sign of the first derivative we could compute the second derivative

$$f''(x) = 3\,(\sin x + \cos x)\,(3 \sin x \cos x - 1)$$

and simply substitute in it the particular values of x. For instance, for $x = 0$ we obtain $f''(0) = -3$ so there is a maximum at this point, while for $x = \pi/4$ we have $f''(\pi/4) = (3/2)\sqrt{2}$, i.e. a minimum, etc.

Let us also determine the abscissae of the points of intersection of the graph with the x-axis, i.e. we solve the equation $\sin^3 x + \cos^3 x = 0$; whence $\cos x = -\sin x$ and therefore $x = 3\pi/4$ or $7\pi/4$.

We now calculate the values of the function corresponding to the determined values of x and we construct the table:

$x =$	0 $(2\pi = 6{,}28)$	$\frac{\pi}{4} = 0.78$	$\frac{\pi}{2} = 1.57$	$\frac{3\pi}{4} = 2{,}36$	$\pi = 3.14$	$\frac{5\pi}{4} = 3.94$	$\frac{3\pi}{2} = 4.71$	$\frac{7\pi}{4} = 5{,}50$
$y =$	1	$\frac{\sqrt{2}}{2} = 0.71$	1	0	-1	$-\frac{\sqrt{2}}{2} = -0.71$	-1	0
	$y' = 0$ max.	$y' = 0$ min.	$y' = 0$ max.		$y' = 0$ min.	$y' = 0$ max.	$y' = 0$ min.	

The graph shown in Fig. 45 has been drawn according to this table.

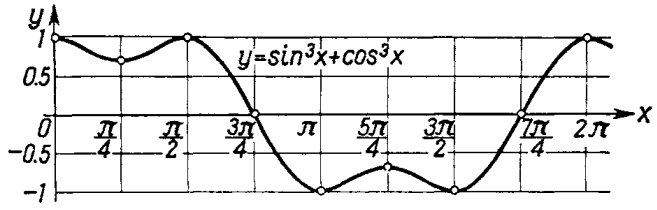

FIG. 45.

(2) Find the extrema and construct the graph of the function

$$f(x) = x^{2/3} - (x^2 - 1)^{1/3}.$$

Now the derivative

$$f'(x) = \frac{2}{3}x^{-1/3} - \frac{1}{3}(x^2-1)^{-2/3}\cdot 2x = \frac{2}{3}\,\frac{(x^2-1)^{2/3} - x^{4/3}}{x^{1/3}\cdot(x^2-1)^{2/3}}$$

exists and is finite everywhere except at the points $x = 0$ and $x = \pm 1$. On approaching these points from the left and from the right the derivative has infinite limits and consequently at these points both two-sided derivatives are infinite [Sec. 103].

To calculate the zeros of the derivative we equate to zero its numerator and we find that $x = \pm 1/\sqrt{2}$. Thus, the points which we suspect may be extrema are the following:

$$-1, \quad -\frac{1}{\sqrt{2}}, \quad 0, \quad \frac{1}{\sqrt{2}}, \quad 1.$$

Incidentally, since the function is even (and consequently its graph is symmetric about the y-axis), it is sufficient to consider the right semi-plane only, i.e. the values $x \geqslant 0$.

For $x = 0$ (and near this point) the numerator and the first factor of the denominator have the plus sign. The factor $x^{1/3}$ however of the denominator changes the minus sign to a plus sign and hence the derivative does the same; thus we are faced with a *minimum*. For $x = 1/\sqrt{2}$ (and near it) the denominator is positive. Now, for the values of x near $1/\sqrt{2}$ we can rewrite the numerator in the form $(1-x^2)^{2/3} - x^{4/3}$; it vanishes for $x = 1/\sqrt{2}$, increases when x decreases and decreases when x increases; consequently its sign changes from plus to minus and so there is maximum at $x = 1/\sqrt{2}$. In passing through the point $x = 1$ the factor $(x^2-1)^{2/3}$ in the denominator, which vanishes at this point, does not change sign. The same is true for the derivative and hence at $x = 1$ there is no extremum.

Although the function is defined and continuous in the whole interval $(-\infty, +\infty)$ it is evident that the construction of the graph can only be accomplished

over a finite interval. However, we can describe the behaviour of the function "at infinity"; writing it in the form

$$f(x) = \frac{1}{x^{4/3} + x^{2/3}(x^2-1)^{1/3} + (x^2-1)^{2/3}},$$

we observe that $f(x) > 0$ and that it tends to zero as $x \to \pm\infty$. Thus the graph of the function is situated above the x-axis and on approaching infinity, both to the left and to the right, the graph tends to this axis.

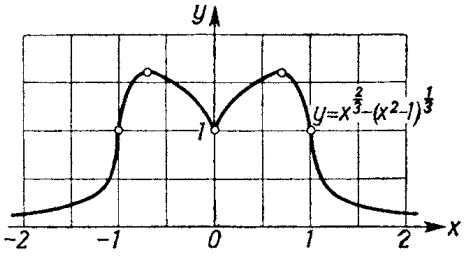

FIG. 46.

The table is as follows:

$x =$	$-\infty$	-1	$-\frac{\sqrt{2}}{2} = -0.71$	0	$\frac{\sqrt{2}}{2} = 0.71$	1	$+\infty$
$y =$	0	1	$\sqrt[3]{4} = 1.59$	1	$\sqrt[3]{4} = 1.59$	1	0
		$y' = +\infty$	$y' = 0$ max.	$y' = \pm\infty$ min.	$y' = 0$ max.	$y' = -\infty$	

and the graph is shown in Fig. 46.

117. Application of higher derivatives. We found that if $f'(x_0) = 0$ and $f''(x_0) > 0$ the function $f(x)$ has a minimum at the point x_0; if now $f'(x_0) = 0$ and $f''(x_0) < 0$ the function has a maximum at this point. The case when $f'(x_0) = 0$ and $f''(x_0) = 0$ was not investigated.

Assume now that in the vicinity of the point $x = x_0$ the function has n successive derivatives and the nth derivative is continuous at the point $x = x_0$. Assume that they all, up to the $(n-1)$th, vanish at the considered point, i.e.

$$f'(x_0) = f''(x_0) = \ldots = f^{(n-1)}(x_0) = 0,$$

and $f^{(n)}(x_0) \neq 0$. Expand the increment $f(x) - f(x_0)$ of the function $f(x)$ into series with respect to the powers of the difference $x - x_0$ by the Taylor formula with the remainder term in Peano's form

[Sec. 107, (17)]. Since all derivatives of orders lower than n vanish at x, we have

$$f(x)-f(x_0)=\frac{f^{(n)}(x_0)+\alpha(x)}{n!}(x-x_0)^n.$$

Since $\alpha\to 0$ as $x\to x_0$, for x sufficiently near x_0 the sign of the sum in the numerator is the same as the sign of $f^{(n)}(x_0)$, for both $x<x_0$ and $x>x_0$. Let us examine the following two cases.

(1) n is an odd number: $n=2k+1$. In passing from the values of x smaller than x_0 to the values greater than x_0 the expression $(x-x_0)^{2k+1}$ changes its sign and since the sign of the first factor is not changed the sign of the difference $f(x)-f(x_0)$ does change. Thus, at point x_0 the function $f(x)$ cannot have an extremum, for near x_0 it takes values both smaller and greater than $f(x_0)$.

(2) n is an even number: $n=2k$. In this case the difference $f(x)-f(x_0)$ does not change sign in passing from x smaller to x greater than x_0, since $(x-x_0)^{2k}>0$ for all x. Obviously, near the point x_0, both on the left and on the right, the sign of the difference is the same as the sign of the number $f^{(n)}(x_0)$. Therefore, if $f^{(n)}(x_0)>0$, then also $f(x)>f(x_0)$ near x_0 and at x_0 the function $f(x)$ has a minimum; similarly, if $f^{(n)}(x_0)<0$ the function has a maximum.

These considerations give us the following rule:

If the first non-vanishing derivative at the point x_0 is a derivative of odd order, then the function has at x_0 neither a maximum nor a minimum. If this derivative is of even order the function has a maximum or minimum at the point x_0, depending on whether this derivative is negative or positive respectively†.

For instance, for the function $f(x)=e^x+e^{-x}+2\cos x$, $x=0$ is a stationary point, since at this point the derivative

$$f'(x)=e^x-e^{-x}-2\sin x$$

vanishes. Furthermore,

$$f''(x)=e^x+e^{-x}-2\cos x, \quad f''(0)=0;$$
$$f'''(x)=e^x-e^{-x}+2\sin x, \quad f'''(0)=0;$$
$$f''''(x)=e^x+e^{-x}+2\cos x, \quad f''''(0)=4.$$

Since the first non-vanishing derivative is of an even order, we are faced with an extremum, namely a minimum, for $f''''(0)>0$.

† This rule was announced in 1742 by Colin Maclaurin in his *A Treatise of Fluxions.*

§ 2. The greatest and the smallest values of a function

118. Determination of the greatest and the smallest values. Suppose that a function $f(x)$ is defined and is continuous in a finite closed interval $[a, b]$. So far, we have been interested in its maxima and minima only; now, however, we shall consider the problem of determining the greatest and smallest of all values which it takes in the considered interval†; according to a known property of continuous functions [Sec. 73], such greatest and smallest values exist. For the sake of definiteness we shall examine the greatest value.

If it occurs at a point between a and b, then it is a maximum as well (obviously the greatest one); however, the greatest value may take place also on one of the ends of the interval, a or b (Fig. 47). Thus we have to compare all maxima of the function $f(x)$ and its boundary values $f(a)$ and $f(b)$; the greatest of these numbers is the greatest of all values of the function $f(x)$ in $[a, b]$. In a similar way we find the smallest value of a function.

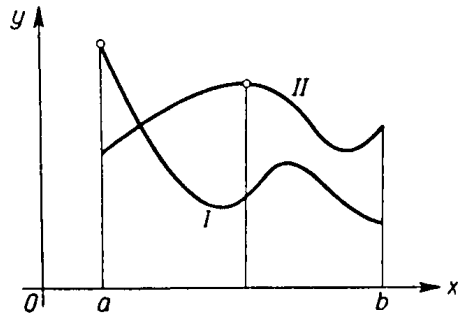

FIG. 47.

If we desire to avoid investigating the maxima or minima, we may use another procedure. We only have to compute the values of the function at all "suspect" points and to compare them with the boundary values $f(a)$ and $f(b)$; it is evident that the greatest and the smallest of these values are the greatest and the smallest of all values of the function.

† Thus we use the term "maximum" in the "local" sense (the greatest value in the immediate neighbourhood of the point), this term being distinct from the greatest value of the function in the whole interval. The same is true for the minimum and the smallest value of a function.

Remark. The case we most frequently encounter in applications is that in which there is only *one* "suspect" point x_0 between a and b. If the function at this point has a maximum (minimum) it is clear, without comparing it with the boundary values, that this is the greatest (smallest) value of the function in the considered interval (Fig. 48). Frequently in similar cases it turns out to be simpler to carry out the investigation of the maximum or the minimum than to compare particular values of the function (particularly if the expression of the function contains literal coefficients).

It is important to emphasize that what we have said above is equally applicable to an *open* interval (a, b), and also to an *infinite* interval.

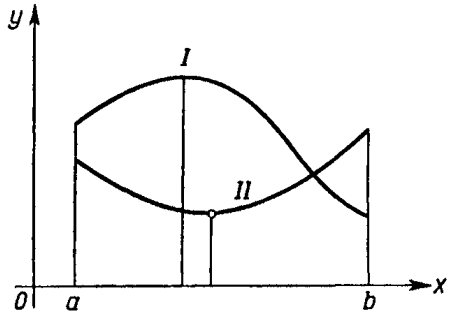

FIG. 48.

119. Problems. We now present a few problems from various fields, the solution of which can be reduced to determining the greatest or the smallest value of a function. Incidentally, usually these values are not so much of interest as the points (the values of the argument) which give the function the considered special values.

(1) We construct a rectangular open box by cutting out of a square sheet of tin with side a, equal squares in the corners and by bending them (Fig. 49). How should we construct the box in order to ensure maximum capacity?

If the side of the cut square be denoted by x, the volume of the box is $y = x(a-2x)^2$, x ranging over the interval $[0, a/2]$. The problem is thus reduced to the determination of the greatest value of the function y in this interval.

Since the derivative $y' = (a-2x)(a-6x)$ has only the one root $x = a/6$ between 0 and $a/2$, so establishing that this value gives a maximum of the function, at the same time we find the required greatest value. In other words, for $x = a/6$ we have $y = 2a^3/27$, while the boundary values of y are equal to zero; consequently, for $x = a/6$ we in fact have the greatest value of y.

(2) A log with circular cross-section of diameter d is given. It is required to cut it in such a way that a beam of rectangular cross-section is obtained with the maximum strength.

Hint. In *Strength of Materials* it is proved that the strength of a rectangular beam is proportional to the product bh^2 where b is the base of the rectangle and h is its height.

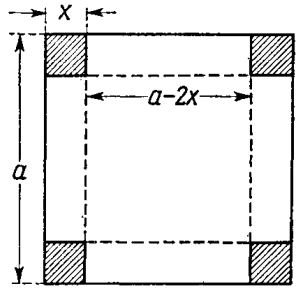

FIG. 49.

Since $h^2 = d^2 - b^2$, we seek the greatest value of the expression $y = bh^2 = b(d^2 - b^2)$, "the independent variable" b ranging over the interval $(0, d)$

The derivative $y' = d^2 - 3b^2$ vanishes only once inside this interval, at the point $b = d/\sqrt{3}$. The second derivative $y'' = -6b > 0$; y therefore takes a maximum value at the above point, which is also the greatest value.

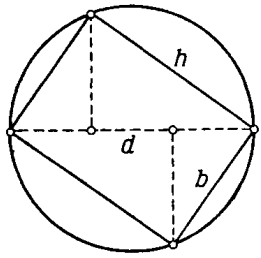

FIG. 50.

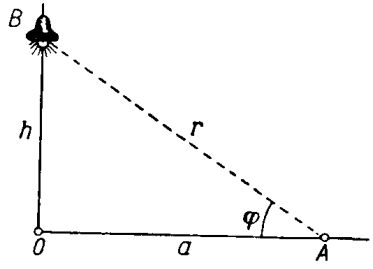

FIG. 51.

For $b = d/\sqrt{3}$ we have $h = d\sqrt{(2/3)}$ and hence $d : h : b = \sqrt{3} : \sqrt{2} : 1$. We can see from Fig. 50 how we can construct the required rectangle—the diameter is divided into three equal parts and perpendiculars are constructed at these points.

(3) Suppose that an electric bulb can move (for instance on a block) along the vertical straight line OB (Fig. 51). At what distance from the horizontal plane OA is it to be located in order that the maximum illumination is obtained at the point A of this plane?

Hint. The illumination J is proportional to $\sin\varphi$ and inversely proportional to the square of the distance $r = AB$, i.e.

$$J = c\frac{\sin\varphi}{r^2},$$

c being independent of the power of the light of the bulb.

If we take as the independent variable $h = OB$, then

$$\sin\varphi = \frac{h}{r}, \quad r = \sqrt{(h^2 + a^2)}$$

and $$J = c\frac{h}{(h^2 + a^2)^{3/2}} \quad (0 < h < +\infty).$$

Further the derivative

$$J_h' = c\frac{a^2 - 2h^2}{(h^2 + a^2)^{5/2}}$$

vanishes for $h = a/\sqrt{2} \doteq 0.7a$, and changes sign from plus to minus in passing through this value of c. This is the most effective distance.

Remark. This is a good opportunity to draw the reader's attention to the following fact. In determining the greatest or the smallest value of a function for a definite interval of the independent variable it may turn out that, inside this interval, there are no roots of the derivative and no other "suspect" values. This implies that in the considered interval the function is monotonically increasing or decreasing and consequently it reaches its greatest and smallest values on the ends of the interval.

§ 3. Solution of indeterminate forms

120. Indeterminate forms of the type 0/0. We shall now employ the concept of the derivative to solve indeterminate forms of all types. First we examine the fundamental case—the indeterminate form of the type 0/0, i.e. we investigate the problem of the limit of the ratio of two functions $f(x)$ and $g(x)$ both of which tend to zero (for instance as $x \to a$). The following theorem was given by John Bernoulli. However, the rule which it contains is usually called l'Hôpital's rule, since it was first (although not in the present form) announced by l'Hôpital† in his book *Analysis of Infinitesimals* published in 1696.

† Guillaume François de l'Hôpital (1661–1704)—a French mathematician. The book quoted was the first printed course of differential calculus.

THEOREM 1. *Suppose that (1) the functions $f(x)$ and $g(x)$ are defined in the interval $(a, b]$; (2) $\lim_{x \to a} f(x) = 0$, $\lim_{x \to a} g(x) = 0$; (3) there exist in the interval $(a, b]$ finite derivatives $f'(x)$ and $g'(x)$ and $g'(x) \neq 0$, and, finally, (4) there exists the limit (finite or otherwise)*

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = K.$$

Then we also have

$$\lim_{x \to a} \frac{f(x)}{g(x)} = K.$$

Proof. We complete the definitions of the functions $f(x)$ and $g(x)$ by assuming that they vanish for $x = a$: $f(a) = g(a) = 0$†. Then these functions become continuous in the whole closed interval $[a, b]$; their values at the point a are identical with the limits as $x \to a$ (by *(2)*), while for the remaining points the continuity follows from the existence of finite derivatives (see *(3)*). Applying the Cauchy theorem [Sec. 104] we obtain

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(c)}{g'(c)},$$

where $a < c < x$. The fact that $g(x) \neq 0$, i.e. $g(x) \neq g(a)$, is a consequence of the assumption that $g'(x) \neq 0$, as was established in proving the Cauchy formula.

Evidently, when $x \to a$ we also have $c \to a$, and hence, in view of *(4)*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{c \to a} \frac{f'(c)}{g'(c)} = K.$$

This completes the proof.

Thus the theorem proved above reduces the limit of the ratio of two functions to the limit of the ratio of their derivatives, provided the latter exists. It frequently happens that the determination of the limit of the ratio of the derivatives is simpler and can be performed by elementary means.

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† We could, of course, simply assume beforehand the functions to be defined and continuous at $x = a$; in applications, however, it is sometimes more convenient to state the theorem used here (for example, see Theorem 1*).

Observe that for definiteness only, we examined the case when a is the left end of the interval and the variable x tends to a from the right. We could assume that a is the right end and the variable x tends to it from the left. Finally, we could examine the two-sided limit process as well.

Examples. (1) Find the limit

$$\lim_{x\to a}\frac{\sqrt{(2a^3x-x^4)}-a\sqrt[3]{(a^2x)}}{a-\sqrt[4]{(ax^3)}}.$$ †

In accordance with the l'Hôpital rule it is equal to the limit

$$\lim_{x\to a}\frac{\dfrac{a^3-2x^3}{\sqrt{(2a^3x-x^4)}}-\dfrac{a^2}{3\sqrt[3]{(ax^2)}}}{-\dfrac{3a}{4\sqrt[4]{(a^3x)}}}=\frac{16}{9}a.$$

The final result is obtained from the ratio of the derivatives by the simple substitution $x=a$, since this ratio is continuous at the considered point.

(2) Find the limit

$$\lim_{x\to 0}\frac{\tan x-x}{x-\sin x}.$$

The ratio of the derivatives is simplified as follows:

$$\frac{\dfrac{1}{\cos^2 x}-1}{1-\cos x}=\frac{1}{\cos^2 x}\,\frac{1-\cos^2 x}{1-\cos x}=\frac{1+\cos x}{\cos^2 x};$$

as $x\to 0$ it of course tends to 2. We obtain the same value for the limit from Theorem 1.

We draw the reader's attention to the fact that here, too, the ratio of the derivatives constitutes an indeterminate form of the type 0/0 but the solution of this indeterminate form was possible by means of elementary transformations. In other cases it may be necessary to apply the theorem once more. It is important to observe that various simplifications of the derived expressions are admissible, e.g. division by common factors, applications of known limits, etc.

Theorem 1 can easily be extended to the case when the argument x tends to the infinite $a=\pm\infty$. For instance, we have then;

THEOREM 1*. *Suppose that (1) the functions $f(x)$ and $g(x)$ are defined in the interval $[c,+\infty)$, (2)* $\lim_{x\to+\infty}f(x)=0$, $\lim_{x\to+\infty}g(x)=0$,

† This is the first example of the solution of an indeterminate form given in l'Hôpital's book.

(3) there exist in the interval $[c, +\infty)$ *finite derivatives* $f'(x)$ *and* $g'(x)$ *and* $g'(x) \neq 0$, *and finally (4) there exists the limit (finite or otherwise)*

$$\lim_{x \to +\infty} \frac{f'(x)}{g'(x)} = K.$$

Then we also have

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = K.$$

Proof. Transform the variable x in accordance with the relation $x = 1/t$, $t = 1/x$. Then, when $x \to +\infty$ we have $t \to +0$, and conversely. In view of *(2)* we have

$$\lim_{t \to +0} f\left(\frac{1}{t}\right) = 0, \quad \lim_{t \to +0} g\left(\frac{1}{t}\right) = 0,$$

and by virtue of *(4)*

$$\lim_{t \to +0} \frac{f'\left(\frac{1}{t}\right)}{g'\left(\frac{1}{t}\right)} = K.$$

We may apply Theorem 1 to the functions $f(1/t)$ and $g(1/t)$ of the new variable t which yields

$$\lim_{t \to +0} \frac{f\left(\frac{1}{t}\right)}{g\left(\frac{1}{t}\right)} = \lim_{t \to +0} \frac{f'\left(\frac{1}{t}\right)\left(-\frac{1}{t^2}\right)}{g'\left(\frac{1}{t}\right)\left(-\frac{1}{t^2}\right)} = \lim_{t \to +0} \frac{f'\left(\frac{1}{t}\right)}{g'\left(\frac{1}{t}\right)} = K^{\dagger}.$$

Then also

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = K.$$

This completes the proof.

121. Indeterminate forms of the type ∞/∞. We now consider indeterminate forms of the type ∞/∞, i.e. we investigate the problem of the limit of the ratio of two functions $f(x)$ and $g(x)$ tending to

† The functions $f(1/t)$ and $g(1/t)$ are differentiated with respect to t as compound functions.

infinity (as $x \to a$). We shall prove that in this case the same rule of l'Hôpital can be applied; the following theorem is just a rewording of Theorem 1.

THEOREM 2. *Suppose that (1) the functions $f(x)$ and $g(x)$ are defined in the interval $(a, b]$; (2) $\lim_{x \to a} f(x) = \infty$, $\lim_{x \to a} g(x) = \infty$, (3) there exist in the interval $(a, b]$ finite derivatives $f'(x)$ and $g'(x)$ and $g'(x) \neq 0$, and finally (4) there exists the limit (finite or otherwise)*

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = K.$$

Then we also have

$$\lim_{x \to a} \frac{f(x)}{g(x)} = K.$$

Proof. In view of *(2)* we may assume that $f(x) > 0$ and $g(x) > 0$ for all values of x.

We first examine the case of finite K. Taking an arbitrary number $\varepsilon > 0$, in accordance with condition (4) a number $\eta > 0$ ($\eta \leqslant b - a$) can be found, such that for $a < x < a + \eta$ we have

$$\left| \frac{f'(x)}{g'(x)} - K \right| < \frac{\varepsilon}{2}.$$

Set, for brevity, $a + \eta = x_0$ and take x between a and x_0. Apply the Cauchy formula† to the interval $[x, x_0]$,

$$\frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(c)}{g'(c)},$$

where $x < c < x_0$; consequently,

$$\left| \frac{f(x) - f(x_0)}{g(x) - g(x_0)} - K \right| < \frac{\varepsilon}{2}. \tag{1}$$

We now write down the identity (the validity of which can be verified directly)

$$\frac{f(x)}{g(x)} - K = \frac{f(x_0) - Kg(x_0)}{g(x)} + \left[1 - \frac{g(x_0)}{g(x)}\right]\left[\frac{f(x) - f(x_0)}{g(x) - g(x_0)} - K\right].$$

† This is the essential difference between this proof and that of Theorem 1: we cannot apply here the Cauchy formula to the interval $[a, x]$, since no matter how we define the functions $f(x)$ and $g(x)$ at a, in view of *(2)* we cannot obtain functions continuous at this point.

Since $g(x) \to \infty$ as $x \to a$, a number $\delta > 0$ (we may assume that $\delta \leqslant \eta$) can be found, such that for $a < x < a + \delta$ we have

$$g(x) > g(x_0) \quad \text{and} \quad \left| \frac{f(x_0) - Kg(x_0)}{g(x)} \right| < \frac{\varepsilon}{2}.$$

For the above indicated values of x (see (1))

$$\left| \frac{f(x)}{g(x)} - K \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which proves the statement.

Suppose now that $K = \infty$ (the case $K = -\infty$ cannot arise because of assumption *(2)*); then $f'(x) \neq 0$ at least for values of x sufficiently near a. Interchanging the roles of the functions f and g we finally obtain

$$\lim_{x \to a} \frac{g'(x)}{f'(x)} = 0,$$

and hence by that which has just been proved above

$$\lim_{x \to a} \frac{g(x)}{f(x)} = 0.$$

Finally, the last relation (in accordance with our remarks) implies that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \infty.$$

In the statement of Theorem 2 we could set $a = -\infty$ with no essential alterations of the proof. If a were the right end of the considered interval, then in particular we could set $a = +\infty$. Thus, the case $a = \pm\infty$ is actually contained in Theorem 2.

Examples.

$$(3) \quad \lim_{x \to +\infty} \frac{\log x}{x^\mu} = \lim_{x \to +\infty} \frac{\frac{1}{x}}{\mu x^{\mu-1}} = \lim_{x \to +\infty} \frac{1}{\mu x^\mu} = 0 \quad (\text{if } \mu > 0),$$

$$(4) \quad \lim_{x \to +\infty} \frac{x^\mu}{a^x} = \lim_{x \to +\infty} \frac{\mu x^{\mu-1}}{a^x \cdot \log a} \quad (a > 1,\ \mu > 0).$$

If $\mu > 1$ we have on the right again an indeterminate form of the same type ∞/∞; however, continuing this process and again applying Theorem 2, we finally obtain in the numerator a power with a negative (or zero) exponent. Hence, in any case

$$\lim_{x \to +\infty} \frac{x^\mu}{a^x} = 0.$$

122. Other types of indeterminate forms. The preceding theorems concerned indeterminate forms of the types $0/0$ and ∞/∞.

If we have an indeterminate form of the type $0\cdot\infty$ it can be reduced to the form $0/0$ or ∞/∞ and then l'Hôpital's rule can be applied. Suppose that

$$\lim_{x\to a} f(x) = 0, \quad \lim_{x\to a} g(x) = \infty$$

and $f(x)$ does not change the sign. Then

$$f(x)\,g(x) = \frac{fx}{\dfrac{1}{g(x)}} = \frac{g(x)}{\dfrac{1}{f(x)}}.$$

The second expression represents an indeterminate form of the type $0/0$ (as $x\to a$), while the third one of the type ∞/∞.

Example (5).

$$\lim_{x\to+0}(x^{\mu}\log x) = \lim_{x\to+0}\frac{\log x}{x^{-\mu}} = \lim_{x\to+0}\frac{\dfrac{1}{x}}{-\mu x^{-\mu-1}} = \lim_{x\to+0}\frac{x^{\mu}}{-\mu} = 0.$$

(we take $\mu>0$).

An indeterminate form of the type $\infty-\infty$ can always be reduced to the form $0/0$ or ∞/∞. Consider the expression $f(x)-g(x)$ where

$$\lim_{x\to a} f(x) = +\infty, \quad \lim_{x\to a} g(x) = +\infty.$$

We can now perform, for instance, the following transformation, reducing this expression to an indeterminate form of the type $0/0$:

$$f(x)-g(x) = \frac{1}{\dfrac{1}{f(x)}} - \frac{1}{\dfrac{1}{g(x)}} = \frac{\dfrac{1}{g(x)} - \dfrac{1}{f(x)}}{\dfrac{1}{f(x)}\,\dfrac{1}{g(x)}}.$$

Incidentally, this can frequently be done in an even simpler way.

Example (6).

$$\lim_{x\to 0}\left(\cot^2 x - \frac{1}{x^2}\right) = \lim_{x\to 0}\frac{x^2\cos^2 x - \sin^2 x}{x^2\sin^2 x},$$

but

$$\frac{x^2\cos^2 x - \sin^2 x}{x^2\sin^2 x} = \frac{x\cos x + \sin x}{x}\,\frac{x\cos x - \sin x}{x\sin^2 x};$$

the limit of the first factor is elementary,

$$\lim_{x\to 0}\frac{x\cos x + \sin x}{x} = \lim_{x\to 0}\left(\cos x + \frac{\sin x}{x}\right) = 2,$$

to the second we apply Theorem 1:

$$\lim_{x\to 0}\frac{x\cos x-\sin x}{x\sin^2 x}=\lim_{x\to 0}\frac{-x\sin x}{\sin^2 x+2x\sin x\cos x}$$

$$=\lim_{x\to 0}\frac{-1}{\frac{\sin x}{x}+2\cos x}=-\frac{1}{3}.$$

Thus, the required limit is $-2/3$.

In the case of indeterminate forms of the type 1^∞, 0^0, ∞^0 it is useful first to find the logarithms of these expressions.

Let $y=[f(x)]^{g(x)}$; then $\log y=g(x)\cdot\log f(x)$. The limit of $\log y$ constitutes an indeterminate form of the examined type $0\cdot\infty$. Suppose that by means of one of the above methods we have found $\lim_{x\to a}\log y$, which turns out to be equal to a finite number k, $+\infty$ or $-\infty$. Then $\lim_{x\to a} y$ is e^k, $+\infty$ or 0, respectively.

Example. (7) Let

$$y=\left(\frac{\sin x}{x}\right)^{\frac{1}{1-\cos x}}.$$

It is required to find $\lim y$ as $x\to 0$ (an indeterminate form of the type 1^∞).

If we assume that $x>0$ (which is permissible since y is an even function), then

$$\log y=\frac{\log\sin x-\log x}{1-\cos x}.$$

Using Theorem 1 we obtain

$$\lim_{x\to 0}\log y=\lim_{x\to 0}\frac{\frac{\cos x}{\sin x}-\frac{1}{x}}{\sin x}=\lim_{x\to 0}\frac{x\cos x-\sin x}{x\sin^2 x}.$$

However, we have just found that this limit is $-1/3$. Hence

$$\lim_{x\to 0} y=e^{-1/3}=\frac{1}{\sqrt[3]{e}}.$$

Remark. Indeterminate forms of the type ∞/∞, $0\cdot\infty$ or $\infty-\infty$ are encountered in the works of Euler; exponential indeterminate forms were introduced by Cauchy. However, none of them gave a strict proof for the case ∞/∞!

CHAPTER 8

FUNCTIONS OF SEVERAL VARIABLES

§ 1. Basic concepts

123. Functional dependence between variables. Examples. We have so far investigated the simultaneous variation of two variables one of which depends on the other: the value of the independent variable fully determined the value of the dependent variable or function. However, there are many cases in which several independent variables occur and to determine the value of the function it is necessary to establish first the values taken *simultaneously* by all of the independent variables.

(1) For instance, the volume V of a circular cylinder is a *function* of radius R of its base and the height H; the dependence between these variables is expressed by the formula

$$V = \pi R^2 H,$$

which makes it possible for us to calculate the values of V corresponding to some known values of the independent variables R and H.

(2) Suppose that the temperature of a mass of gas contained under the piston in a cylinder is not constant; then the volume V and the pressure p of the mass of gas are connected with its (absolute) temperature T by the so-called Clapeyron formula

$$pV = RT \quad (R = \text{const}).$$

Consequently, regarding for instance V and T as independent variables, we can express the function p by the formula

$$p = \frac{RT}{V}.$$

(3) Investigating the physical state of a body it is frequently necessary to observe the change of its properties with position. For instance density, temperature, electric potential are all functions of position. All these quantities are therefore functions of the coordinates x, y, z of the position of the points of the body. If the physical state of the body is variable in time, then the time t must be added to the above independent variables. In this case we have a function of four independent variables.

The reader can easily construct further examples.

In framing a more precise definition of the concept of a function in the case of several variables we begin with the simplest case of two variables.

124. Functions of two variables and their domains of definition. When speaking of the variation of two independent variables we have to state the pairs of values (x, y) which they may simultaneously take; the set $\mathcal{M}$ of these pairs is the domain of variation of the variables x, y.

The definition of the concept of function is given in the same terms as in the case of a function of one independent variable.

A variable z (with the domain of variation $\mathcal{E}$) is called a function of the independent variables x, y in the set $\mathcal{M}$ if, in accordance with a rule or a law, to every pair (x, y) of their values from $\mathcal{M}$ there corresponds one definite value of z (in $\mathcal{E}$).

We are considering a one-valued function; the definition can easily be extended to the case of a many-valued function.

The set $\mathcal{M}$ is called the *domain of definition* of the function. The variables x, y themselves are called the arguments of the function z. The functional dependence between z, x, y is denoted, as in the case of one variable, as follows:

$$z = f(x, y), \quad z = \varphi(x, y), \quad z = z(x, y), \quad \text{etc.}$$

If a pair (x_0, y_0) is taken from $\mathcal{M}$, then $f(x_0, y_0)$ is the particular (numerical) value of the function $f(x, y)$ when $x = x_0$, $y = y_0$.

We give a few examples of functions defined analytically, i.e. by formulae, the domains of definition being indicated. The formula

$$(1) \quad z = x^2 + y^2$$

defines a function for all pairs (x, y), without exception. The formulae

$$(2) \quad z = \sqrt{(1 - x^2 - y^2)}, \quad (3) \quad z = \frac{1}{\sqrt{(1 - x^2 - y^2)}}$$

are valid (if we consider only finite real values of z) only for the pairs (x, y) which satisfy the inequalities

$$x^2 + y^2 \leqslant 1 \quad \text{or} \quad x^2 + y^2 < 1$$

respectively.

The formula

$$(4) \quad z = \arcsin \frac{x}{a} + \arcsin \frac{y}{b}$$

defines a function for the values of x and y which satisfy the inequalities

$$-a \leqslant x \leqslant a, \quad -b \leqslant y \leqslant b.$$

In all these cases we have indicated the widest natural [Sec. 18, *(2)*] domain over which the formulae can hold.

Consider now the following example.

(5) Suppose that the sides of a triangle vary in an arbitrary manner, the only restriction being that the perimeter is constant and equal to $2p$. If two sides are denoted by x and y, the third side is $2p-x-y$; consequently the triangle is fully determined by the sides x and y. How does the area z depend on these variables?

According to a well-known formula the area has the form

$$z = \sqrt{[p(p-x)(p-y)(x+y-p)]}.$$

The domain of definition $\mathcal{M}$ of this function is now determined by the problem which led us to consider the function. Since the length of each side is a positive number smaller than the perimeter, we have the inequalities

$$0 < x < p, \quad 0 < y < p, \quad x+y > p;$$

they describe the domain $\mathcal{M}$ although the derived analytic expression is meaningful in a wider domain, for instance for $x > p$ and $y > p$.

Thus, whereas for a function of one variable the standard domain of variation of the argument was an interval, in the case of a function of two independent variables there is a greater variety of the possible (and natural) domains of variation of the arguments.

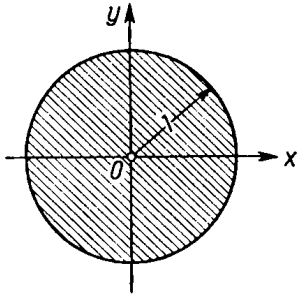

FIG. 52.

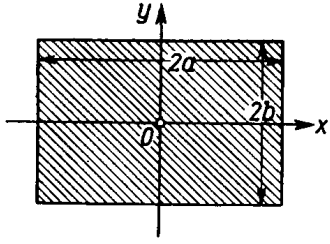

FIG. 53.

Investigation of the domains is considerably simplified by considering their geometric interpretation. If we construct on the plane two perpendicular axes and, as usual, introduce the values of x and y, then every pair (x, y) uniquely defines a point on the plane, the considered values being the coordinates of the point. The converse is also true.

Thus to describe the pairs (x, y) for which the function is defined it is simply necessary to indicate the figure on the xy plane covered by the corresponding points.

For instance, we say that function *(1)* is defined in the entire plane, functions *(2)* and *(3)* in the circles—a closed one (including the circumference) and an open one, respectively (Fig. 52); the function *(4)* is defined in the rectangle (Fig. 53); finally function *(5)* is defined in an open triangle (Fig. 54).

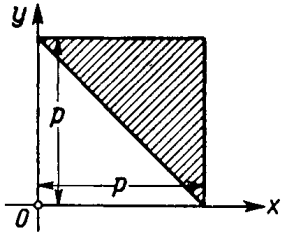

FIG. 54.

The geometric interpretation is so convenient that usually the pairs of values (x, y) themselves are called "points" and the set of such "points" corresponding to some geometric image is called by the same name as the image. Thus the set of "points" or pairs (x, y) for which the inequalities

$$a \leqslant x \leqslant b, \quad c \leqslant y \leqslant d,$$

hold is a "rectangle" the dimensions of which are $b-a$ and $d-c$; it will be denoted by the symbol $[a, b; c, d]$, just as for an interval. The set of "points" or pairs (x, y) satisfying the inequality

$$(x-\alpha)^2+(y-\beta)^2 \leqslant r^2$$

is a "circle" of radius r with centre at the "point" (α, β), and so on.

Just as for the geometric illustration of the function $y=f(x)$ by its graph [Sec. 19] we can interpret geometrically the equation $z=f(x, y)$. Take in space a rectangular system of coordinate axes x, y, z and indicate on the xy plane the domain $\mathcal{M}$ of variation of the variables x and y; finally at every point $M(x, y)$ of this domain construct a perpendicular to the xy plane and measure on it the value $z=f(x, y)$. The geometric image of the points constructed in this way is a kind of spatial graph of our function. In general, it is a surface; consequently the relation $z=f(x, y)$ is called the equation of the surface.

As examples, Figs. 55 and 56 represent the geometric images of the functions

$$z = x^2 + y^2 \quad \text{and} \quad z = \sqrt{(1 - x^2 - y^2)}.$$

The first is a paraboloid of revolution while the second is a hemisphere.

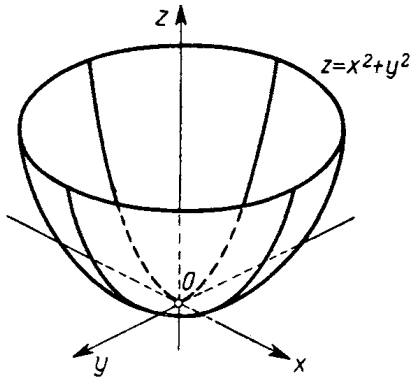

FIG. 55.

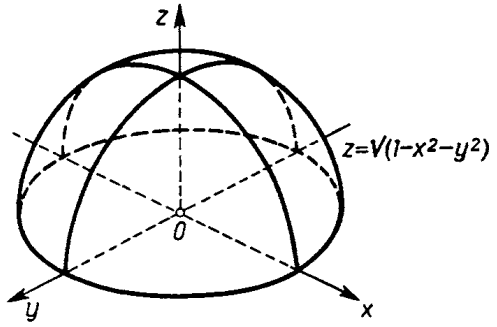

FIG. 56.

125. Arithmetic m-dimensional space. We now consider functions of m independent variables (for $m \geqslant 3$); we first examine the systems of simultaneous values of these variables.

In the case $m = 3$, such a system consists of three numbers (x, y, z) and it is clear that it can be interpreted geometrically as a point of space, and the set of such systems of three numbers as a part of the space or a geometric body; but for $m > 3$ we can no longer give a direct geometric interpretation.

Nevertheless, we aim at extending the geometric methods (which turned out to be so effective for functions of two and three variables) to the theory of functions of a greater number of variables, so we introduce into analysis a concept of m-dimensional "space", where $m > 3$.

We shall define an m-dimensional "point" by a system of m real numbers: $M(x_1, x_2, \ldots, x_m)$†; the numbers $x_1, x_2, \ldots, x_m$ are the coordinates of this "point" M. The set of all possible m-dimensional "points" constitutes an m-dimensional "space" which is sometimes called "arithmetic".

The concept of the "m-dimensional point" and the "m-dimensional (arithmetic) space" is due to Riemann‡ but the terminology is due to Cantor.

It is expedient to introduce the concept of "distance" $\overline{MM'}$ between two m-dimensional "points":

$$M(x_1, x_2, \ldots, x_m) \quad \text{and} \quad M'(x_1', x_2', \ldots, x_m').$$

Corresponding to the familiar formula of analytic geometry we set

$$\overline{MM'} = \overline{M'M} = \sqrt{\left[\sum_{i=1}^{m} (x_i' - x_i)^2\right]}$$
$$= \sqrt{[(x_1' - x_1)^2 + (x_2' - x_2)^2 + \ldots + (x_m' - x_m)^2]}; \qquad (1)$$

for $m = 2$ or 3 this "distance" is identical to the ordinary distance between two geometric points.

If we take one more point

$$M''(x_1'', x_2'', \ldots, x_m''),$$

it can be proved that for the "distances" $\overline{MM'}, \overline{M'M''}, \overline{MM''}$ the following inequality holds:

$$\overline{MM''} \leqslant \overline{MM'} + \overline{M'M''}; \qquad (2)$$

it resembles the familiar theorem of geometry: "Any side of a triangle is not greater than the sum of the two remaining sides."

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† When dealing with an indefinite number of unknowns it is convenient to denote them not by different letters but by the same letter with various indices. Thus x_i denotes (contrary to the previous notation) not the ith value of a variable but the ith variable itself, which independently takes various values.

‡ Bernhard Riemann (1826–1866)—an outstanding German mathematician.

In fact, for any set of real numbers $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_m$ we have the inequality

$$\sqrt{\left[\sum_{i=1}^{m}(a_i+b_i)^2\right]} \leqslant \sqrt{\left(\sum_{i=1}^{m}a_i^2\right)} + \sqrt{\left(\sum_{i=1}^{m}b_i^2\right)}.^{\dagger}$$

Setting

$$a_i = x_i' - x_i, \quad b_i = x_i'' - x_i', \quad \text{so} \quad a_i + b_i = x_i'' - x_i,$$

we obtain

$$\sqrt{\left[\sum_{i=1}^{m}(x_i''-x_i)^2\right]} \leqslant \sqrt{\left[\sum_{i=1}^{m}(x_i'-x_i)^2\right]} + \sqrt{\left[\sum_{i=1}^{m}(x_i''-x_i')^2\right]},$$

which is equivalent to (2). Thus this essential property of the distance occurs in the new "space" as well.

In an m-dimensional "space" we may also consider "straight lines". The reader may remember that on the $x_1 x_2$ plane the straight line is defined by the equation $(x_1-\beta_1)/\alpha_1 = (x_2-\beta_2)/\alpha_2$, while in the $x_1 x_2 x_3$ space by the equations $(x_1-\beta_1)/\alpha_1 = (x_2-\beta_2)/\alpha_2 = (x_3-\beta_3)/\alpha_3$ (the α coefficients cannot vanish simultaneously). Analogously, we understand by a "straight line" in an m-dimensional "space" the set of "points" $(x_1, x_2, \ldots, x_m)$ which satisfy the system of equations

$$\frac{x_1-\beta_1}{\alpha_1} = \frac{x_2-\beta_2}{\alpha_2} = \ldots = \frac{x_m-\beta_m}{\alpha_m}$$

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† Squaring both sides and omitting in both sides equal terms we reduce this inequality to the familiar Cauchy inequality

$$\sum_{i=1}^{m} a_i b_i \leqslant \sqrt{\left(\sum_{i=1}^{m}a_i^2\right)} \cdot \sqrt{\left(\sum_{i=1}^{m}b_i^2\right)}.$$

We can, incidentally, prove that the latter can be derived in an elementary way. The quadratic expression

$$\sum_{i=1}^{m}(a_i x + b_i)^2 = x^2 \cdot \sum_{i=1}^{m} a_i^2 + 2x \cdot \sum_{i=1}^{m} a_i b_i + \sum_{i=1}^{m} b_i^2$$

does not take negative values. Hence it cannot have different real roots and

$$\sum_{i=1}^{m} a_i^2 \cdot \sum_{i=1}^{m} b_i^2 - \left(\sum_{i=1}^{m} a_i b_i\right)^2 > 0,$$

which is equivalent to the Cauchy inequality.

(bearing in mind the previous condition of α). If we denote the common value of these ratios by t, we can define the "straight line" by the parametric equations

$$x_1 = \alpha_1 t + \beta_1,\ x_2 = \alpha_2 t + \beta_2, \ldots, x_m = \alpha_m t + \beta_m,$$

where the parameter t varies between $-\infty$ and $+\infty$. We regard the "points" as following each other in the order of increasing parameter; if $t' < t < t''$ the "point" M of the "points" M', M, M'' lies between the two other points, since it follows M' and precedes M''. Under these conditions it is easy to prove that the distances between them satisfy the relation

$$\overline{M'M''} = \overline{M'M} + \overline{MM''},$$

which is characteristic of the straight line in ordinary space.

The equation of the "straight line" passing through two known "points"

$$M'(x_1', \ldots, x_m') \quad \text{and} \quad M''(x_1'', \ldots, x_m'')$$

can evidently be written in the form

$$x_1 = x_1' + t(x_1'' - x_1'), \quad \ldots, \quad x_m = x_m' + t(x_m'' - x_m')$$
$$(-\infty < t < +\infty),$$

the "points" M' and M'' being obtained by setting $t = 0$ and $t = 1$. As t varies from zero to unity we obtain "the segment of straight line" $M'M''$ connecting the considered points.

Finally, adjacent "segments" $M'M_1, M_1M_2, \ldots, M_kM''$ constitute a broken line in the space.

126. Examples of domains in m-dimensional space. We now proceed to consider the simplest "bodies" or "domains" in m-dimensional "space".

(1) The set of "points" $M(x_1, x_2, \ldots, x_m)$ the coordinates of which satisfy independently the inequalities

$$a_1 \leqslant x_1 \leqslant b_1,\ \ a_2 \leqslant x_2 \leqslant b_2,\ \ \ldots,\ \ a_m \leqslant x_m \leqslant b_m,$$

is called the m-dimensional "rectangular parallelepiped" and is denoted as follows:

$$[a_1, b_1; a_2, b_2; \ldots; a_m, b_m].$$

For $n = 2$ we obtain, in particular, the "rectangle" considered already in Sec. 124; to the three-dimensional "parallelepiped" there corresponds in space the ordinary rectangular parallelepiped.

If we exclude the equality sign we have

$$a_1 < x_1 < b_1,\ a_2 < x_2 < b_2,\ \ldots,\ a_m < x_m < b_m,$$

thus defining an open "rectangular parallelepiped"

$$(a_1,\ b_1;\ a_2,\ b_2;\ \ldots;\ a_m,\ b_m);$$

in order to distinguish between this and the one defined by the previous relations, the latter is called "closed". The differences $b_1 - a_1$, $b_2 - a_2, \ldots, b_m - a_m$ are called the dimensions of the parallelepipeds, while the point

$$\left(\frac{a_1 + b_1}{2},\ \frac{a_2 + b_2}{2},\ \ldots,\ \frac{a_m + b_m}{2}\right)$$

is called their centre.

By a neighbourhood of the "point" $M_0(x_1^0, x_2^0, \ldots, x_m^0)$ we understand any open "parallelepiped"

$$(x_1^0 - \delta_1,\ x_1^0 + \delta_1;\ x_2^0 - \delta_2,\ x_2^0 + \delta_2;\ \ldots;\ x_m^0 - \delta_m,\ x_m^0 + \delta_m) \quad (3)$$

$(\delta_1, \delta_2, \ldots, \delta_m > 0)$ with centre at the "point" M_0; often it is a "cube"

$$(x_1^0 - \delta,\ x_1^0 + \delta;\ x_2^0 - \delta,\ x_2^0 + \delta;\ \ldots;\ x_m^0 - \delta,\ x_m^0 + \delta)$$

$(\delta > 0)$, all the dimensions of which are equal $(= 2\delta)$.

(2) Consider the set of "points" $M(x_1, x_2, \ldots, x_m)$ which satisfy the inequality

$$(x_1 - x_1^0)^2 + (x_2 - x_2^0)^2 + \ldots + (x_m - x_m^0)^2 \leqslant r^2 \quad (\text{or } < r^2),$$

where $M_0(x_1^0, x_2^0, \ldots, x_m^0)$ is a fixed "point" and r a constant positive number. This set is called a closed (or open) m-dimensional "sphere" of radius r, with centre at the "point" M_0. In other words, the "sphere" is a set of points M, "the distance" of which from a fixed point M_0 does not exceed (or is smaller than) r. It is clear that this "sphere" is a circle when $m = 2$ [cf. Sec. 124], and it is the ordinary sphere when $m = 3$.

An open "sphere" of an arbitrary "radius" $r > 0$ with centre and the "point" $M_0(x_1^0, x_2^0, \ldots, x_m^0)$ may also be regarded as a

neighbourhood of this point; in contrast to the "parallelepipedal" neighbourhood introduced above, this neighbourhood will be called "spherical".

It is useful to realize once for all that, if a "point" M_0 is surrounded by a neighbourhood of one of the two types indicated, it can also be surrounded by a neighbourhood of the other type in such a way that the latter neighbourhood is contained in the former one.

Consider first the "parallelepiped" (3) with centre at the "point" M_0. It is sufficient to take an open "sphere" with the same centre and radius r smaller than all $\delta_i (i = 1, 2, \ldots, m)$, in order that this sphere be contained in the considered "parallelepiped". In fact, for any "point" $M(x_1, x_2, \ldots, x_m)$ of this "sphere" we have (for every i)

$$|x_1 - x_i^0| \leqslant \sqrt{\left[\sum_{k=1}^{m} (x_k - x_k^0)^2\right]} = \overline{MM_0} < r < \delta_i$$

or

$$x_i^0 - \delta_i < x_i < x_i^0 + \delta_i,$$

and therefore this point belongs to the given parallelepiped.

Conversely, if a "sphere" of radius r and centre M_0 is given, then "the parallelepiped" (3) is contained in it when, for instance, $\delta_1 = \delta_2 = \ldots = \delta_m = r/\sqrt{m}$. This fact follows from the fact that any "point" $M(x_1, x_2, \ldots, x_m)$ of this "parallelepiped" is at the "distance"

$$\overline{MM_0} = \sqrt{\left[\sum_{k=1}^{m} (x_k - x_k^0)^2\right]} < \sqrt{\left(\sum_{k=1}^{m} \delta_k^2\right)} = r$$

from the "point" M. Consequently it belongs to the given "sphere".

127. General definition of open and closed domains. We call the "point" $M'(x_1', x_2', \ldots, x_m')$ an *interior* "point" of the set $\mathcal{M}$ in an m-dimensional "space", if this point together with a sufficiently small neighbourhood of it belongs to the set $\mathcal{M}$. It follows from the proposition proved in the preceding section that the type of neighbourhood is irrelevant—i.e. whether it is "parallelepipedal" or "spherical".

For an open "rectangular parallelepiped"

$$(a_1, b_1, \ldots; a_m, b_m) \tag{4}$$

every "point" belonging to it is interior. In fact, if

$$a_1 < x_1' < b_1, \ldots, a_m < x_m' < b_m,$$

it is easy to find $\delta > 0$ such that

$$a_1 < x_1' - \delta < x_1' + \delta < b_1, \ldots, a_m < x_m' - \delta < x_m' + \delta < b_m.$$

Similarly in the case of an open "sphere" of radius r with centre at "point" M_0, every point M' belonging to it is also interior. If we take ϱ such that

$$0 < \varrho < r - \overline{M'M_0},$$

and describe about M' "a sphere" of radius ϱ then it is wholly contained in the original "sphere": provided $\overline{MM'} < \varrho$, we have at once [Sec. 125, (2)]

$$\overline{MM_0} \leqslant \overline{MM'} + \overline{M'M_0} < \varrho + \overline{M'M_0} < r,$$

and hence the "point" M belongs to the original "sphere".

Such a set consisting of interior "points" only will be called an open "domain".

Thus, the open "rectangular parallelepiped" and the open "sphere" are examples of open "domains".

We shall now generalize the concept of a point of condensation [Sec. 32] to the case of a set $\mathcal{M}$ in an m-dimensional "space". A "point" M_0 is called a "point of condensation" of the set $\mathcal{M}$ if in every neighbourhood of this point (the type is again irrelevant) there lies at least one "point" of the set $\mathcal{M}$ distinct from M_0.

"Condensation points" of an open "domain" which do not belong to the domain itself are called the boundary "points" of this "domain". The set of boundary "points" constitute "the boundary of the domain". An open "domain" completed by its "boundary" is called a closed "domain".

It is readily seen that for an open parallelepiped (4) the boundary points are the "points" $M(x_1, x_2, \ldots, x_m)$ for which

$$a_1 \leqslant x_1 \leqslant b_1, \ \ldots, \ a_m \leqslant x_m \leqslant b_m,$$

and in at least one case the equality occurs.

Similarly, for the above open "sphere" the boundary "points" are the "points" M such that $\overline{MM_0} = r$.

Thus the closed "rectangular parallelepiped" and the closed "sphere" are examples of closed "domains".

Henceforth, speaking of a "domain", open or closed, we shall always mean "domain" in the special sense given here.

We now proceed to establish that a closed "domain" contains all its "points of condensation".

Consider a closed "domain" $\overline{\mathcal{D}}$ and a "point" M_0 outside it. We shall then prove that M_0 cannot be a "point of condensation" of $\overline{\mathcal{D}}$.

A closed "domain" $\overline{\mathcal{D}}$ is obtained from an open "domain" $\mathcal{D}$ by joining to the latter its "boundary" $\mathcal{C}$. Clearly, M_0 is not a "point of condensation" of $\mathcal{D}$; consequently M_0 can be surrounded by an open "sphere" which does not contain any "points" of $\mathcal{D}$. But then there can be no "points" from $\mathcal{C}$ in it either: for, if a "point" M' from $\mathcal{C}$ belonged to the sphere, it would also contain a neighbourhood of the "point" M' and this would then contain no point from $\mathcal{D}$, contrary to the definition of the "boundary". Thus, in the considered "sphere" there are no "points" of $\overline{\mathcal{D}}$. This completes the proof.

In general a "point set" $\mathcal{M}$ containing all its "points of condensation" is said to be *closed.* Thus a closed "domain" is a particular case of a closed set.

All the results given in the preceding sections can be regarded as establishing a geometric language†; it is not connected (for $m > 3$) with any real geometric concepts. However, it is useful to note that in fact the m-dimensional "space" is only the first step towards some very fruitful generalizations of the concept of space; these constitute the foundation of many advanced parts of modern higher analysis.

128. Function of m variables. Consider m variables $x_1, x_2, \ldots, x_m$ the simultaneous values of which can be taken arbitrarily over

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† We have enclosed in inverted commas all geometric terms which have been employed in a sense distinct from the ordinary: "point", "distance", "domain", etc. Henceforth we shall stop doing so.

a set $\mathcal{M}$ of points of m-dimensional space: these variables are called *independent*. The definition of a function, and all we have said in this connection for the case of two variables [Sec. 124], can directly be extended to the present case; therefore we shall not repeat the discussion.

If a point $(x_1, x_2, \ldots, x_m)$ is denoted by M, the function $u = f(x_1, \ldots, x_m)$ of these variables is sometimes called the function of the point M and denoted by the symbol $u = f(M)$.

Assume now that in a set $\mathcal{P}$ of points of a k-dimensional space (k is independent of m), m functions of the k variables $t_1, t_2, \ldots, t_k$ are given:

$$x_1 = \varphi_1(t_1, t_2, \ldots, t_k), \ldots, x_m = \varphi_m(t_1, t_2, \ldots, t_k), \tag{5}$$

or, briefly,

$$x_1 = \varphi_1(P), \ldots, x_m = \varphi_m(P), \tag{5a}$$

P denoting the point $(t_1, t_2, \ldots, t_k)$ of the k-dimensional space. Moreover, we assume that when the point P $(t_1, t_2, \ldots, t_k)$ varies over the set $\mathcal{P}$, the corresponding m-point M with coordinates (5) or (5a) at all times belongs to the m-dimensional set $\mathcal{M}$ over which the function $u = f(x_1, \ldots, x_m) = f(M)$ is defined.

Then the variable u can be regarded as a compound function of the independent variables $t_1, t_2, \ldots, t_k$ (over the set $\mathcal{P}$) by means of the variables $x_1 x_2, \ldots, x_m$:

$$u = f[\varphi_1(t_1, t_2, \ldots, t_k), \ldots, \varphi_m(t_1, t_2, \ldots, t_k)];$$

and it is a function of the functions $\varphi_1, \ldots, \varphi_m$ [cf. Sec. 25].

The process of defining a compound function in terms of the functions $\varphi_1, \ldots, \varphi_m$ and the function f is called (as for the simplest case of functions of one variable) *superposition.*

The class of functions of several variables we initially consider is very small. It is virtually constructed by superposition from elementary functions of one variable [Secs. 22, 24] and the following functions of two variables:

$$z = x \pm y, \quad z = xy, \quad z = \frac{x}{y}, \quad z = x^y,$$

i.e. the four arithmetic operations and the so-called power-exponential function.

Arithmetic operations applied again to the independent variables $x_1, x_2, \ldots, x_m$, and to constants, lead first of all to the polynomials

$$P(x_1, x_2, \ldots, x_m) = \sum_{\nu_1, \nu_2, \ldots, \nu_m} C_{\nu_1, \nu_2, \ldots, \nu_m} x_1^{\nu_1} x_2^{\nu_2}, \ldots, x_m^{\nu_m}\dagger \tag{6}$$

(an integral rational function) and to the quotient of such polynomials

$$Q(x_1, x_2, \ldots, x_m) = \frac{\sum C_{\nu_1, \nu_2, \ldots, \nu_m} x_1^{\nu_1} x_2^{\nu_2}, \ldots, x_m^{\nu_m}}{\sum C'_{\mu_1, \mu_2, \ldots, \mu_m} x_1^{\mu_1} x_2^{\mu_2}, \ldots, x_m^{\mu_m}} \tag{7}$$

(a fractional rational function).

Introducing elementary functions of one variable leads, for instance, to the following functions:

$$f(x, y, z) = \frac{\log(x+y+z)}{\sqrt{(x^2+y^2+z^2)}},$$

$$\varphi(x, y, z, t) = \sin xy + \sin yz + \sin zt + \sin tx,$$

etc.

The remarks in Sec. 18 concerning the analytic definition of a function of one variable also apply in the present case.

129. Limit of a function of several variables. Consider the sequence of points

$$\{M_n(x_1^{(n)}, x_2^{(n)}, \ldots, x_m^{(n)})\} \quad (n = 1, 2, 3, \ldots) \tag{8}$$

in m-dimensional space. We say that this sequence converges to the limit point $M_0(a_1, a_2, \ldots, a_m)$ if the coordinates of the point M_n converge separately to the corresponding coordinates of the point M_0, i.e. as $n \to \infty$ we have

$$x_1^{(n)} \to a_1, \quad x_2^{(n)} \to a_2, \quad \ldots, \quad x_m^{(n)} \to a_m. \tag{9}$$

Alternatively we could require that the distance between the points M_n and M_0 tends to zero

$$\overline{M_0 M_n} \to 0. \tag{10}$$

The equivalence of the two definitions follows from the proposition proved in Sec. 126 concerning the neighbourhoods of the

† We have previously used the sign $\sum$ to denote the sum of terms of one variable index. We use it here in the more general case where the terms depend on several indices.

two types. In fact, condition (9) means that for an arbitrary number $\delta > 0$, for a sufficiently large n, the point M_n satisfies the inequalities

$$|x_1^{(n)} - a_1| < \delta, \quad \ldots, \quad |x_m^{(n)} - a_m| < \delta,$$

i.e. it belongs to the open parallelepiped

$$(a_1 - \delta,\ a_1 + \delta,\ \ldots,\ a_m - \delta,\ a_m + \delta)$$

with centre at the point M_0; now, by the requirement (10), for an arbitrary number $r > 0$, the point M_n—again for a sufficiently large n—satisfies the inequality

$$\overline{M_0 M_n} < r,$$

i.e. it lies in the open sphere of radius r with centre at the considered point.

Consider a set $\mathcal{M}$ in m-dimensional space, the point M_0 $(a_1, a_2, \ldots, a_m)$ being a point of condensation of it. Then we can extract from $\mathcal{M}$ a sequence (8) of points distinct from M_0 which has M_0 as the limit point.

Assume now that a function $f(x_1, \ldots, x_m)$ is defined over the introduced set. Similarly, in the case of a function of one variable we say that:

The function $f(x_1, \ldots, x_m) = f(M)$ has for its limit the number A when the variables $x_1, x_2, \ldots, x_m$ tend to $a_1, a_2, \ldots, a_m$, respectively (or, briefly, when the point M tends to the point M_0) if for any sequence (8) *of points from $\mathcal{M}$ distinct from $M_0(a_1, a_2, \ldots, a_m)$, but converging to M_0, the numerical sequence $\{f(x_1^{(n)}, \ldots, x_m^{(n)})\}$ consisting of the corresponding values of the function, always converges to A.*

This is written as follows:

$$A = \lim_{\substack{x_1 \to a_1 \\ \cdots \\ x_m \to a_m}} f(x_1, \ldots, x_m),$$

or, briefly,

$$A = \lim_{M \to M_0} f(M).$$

The definition of the limit of a function can easily be extended to the case when some, or all, of the numbers $A, a_1, \ldots, a_m$ are infinite.

We emphasize that for functions of several variables the concept of the limit of a function reduces to the concept of the limit of a sequence.

However, again the definition of the limit can be presented in the "ε–δ language" without introducing sequences. For finite numbers $A, a_1, \ldots, a_m$ the appropriate definition is as follows:

The function $f(x_1, \ldots, x_m)$ has for its limit the number A as the variables $x_1, x_2, \ldots, x_m$ tend to $a_1, a_2, \ldots, a_m$, respectively, when for any number $\varepsilon > 0$ a number $\delta > 0$ can be found, such that

$$|f(x_1, \ldots, x_m) - A| < \varepsilon,$$

provided

$$|x_1 - a_1| < \delta, \quad \ldots, \quad |x_m - a_m| < \delta.$$

The point $(x_1, \ldots, x_m)$ belongs to $\mathcal{M}$ and is distinct from $(a_1, \ldots, a_m)$. Thus, the inequality should hold for the function at all points of the set $\mathcal{M}$ lying in a sufficiently small neighbourhood

$$(a_1 - \delta,\ a_1 + \delta\,;\ \ldots\,;\ a_m - \delta,\ a_m + \delta)$$

of the point M_0, excluding this point (even when it belongs to $\mathcal{M}$).

In geometric language, writing for the points $(x_1, \ldots, x_m)$ and $(a_1, \ldots, a_m)$ the symbols M and M_0 we could state the result as follows: *the number A is called the limit of the function $f(M)$ when the point M tends to the point M_0 (or is called the limit at the point M_0) if for any number $\varepsilon > 0$ there exists a number $r > 0$, such that*

$$|f(M) - A| < \varepsilon,$$

provided the distance $\overline{M_0M} < r$.

As before, it is assumed that the point M belongs to $\mathcal{M}$ but is distinct from M_0. Thus the inequality for the function must be satisfied at all points of the set $\mathcal{M}$ lying in a small spherical neighbourhood of M_0, excluding the point M_0 itself.

The remark of Sec. 126 on neighbourhoods of various types immediately implies the equivalence of the two forms of the new definition of the limit of function.

The equivalence of the new definition and the former one in the "language of sequences" can be established as in the case of the function of one variable [Sec. 33].

We should observe finally that the whole theory of limits developed above (Chapter III) can be extended to the general case of functions of several variables. In the main, this extension follows automatically since, in the present case, everything can be reduced to the consideration of a sequence [cf. Sec. 42].

130. Examples. *(1)* Making use of the theorem on the limit of a product it is easy to prove that

$$\lim_{\substack{x_1 \to a_1 \\ \cdots \\ x_m \to a_m}} C x_1^{\nu_1} \ldots x_m^{\nu_m} = C a_1^{\nu_1} \ldots a_m^{\nu_m},$$

where C, a_1, a_2, ..., a_m are arbitrary real numbers and ν_1, ..., ν_m are non-negative integers. Hence, denoting by $P(x_1, \ldots, x_m)$ the integral rational function (6) we have, by the theorem on the limit of sum,

$$\lim_{\substack{x_1 \to a_1 \\ \cdots \\ x_m \to a_m}} P(x_1, \ldots, x_m) = P(a_1, \ldots, a_m).$$

Similarly, for a fractional rational function (7), according to the theorem on the limit of a quotient, we obtain

$$\lim_{\substack{x_1 \to a_1 \\ \cdots \\ x_m \to a_m}} Q(x_1, \ldots, x_m) = Q(a_1, \ldots, a_m),$$

of course, provided that the denominator does not vanish at the point $(a_1, a_2 \ldots, a_m)$.

(2) Consider the power-exponential function x^y for $x > 0$ and an arbitrary y. Then, if $a > 0$ and b is an arbitrary real number, we have

$$\lim_{\substack{x \to a \\ y \to b}} x^y = a^b.$$

In fact, taking any variables depending on n, $x_n \to a$ and $y_n \to b$ we have [see Sec. 66]

$$x_n^{y_n} = e^{y_n \log x_n} \to e^{b \log a} = a^b,$$

and this establishes the required result in the "language of sequences".

(3) Consider the problem of the limit

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{xy}{x^2 + y^2};$$

this function is defined over the whole plane except for the point $x = 0$, $y = 0$.

On taking two partial sequences of points

$$\left\{M_n\left(\frac{1}{n}, \frac{1}{n}\right)\right\} \quad \text{and} \quad \left\{M_n'\left(\frac{2}{n}, \frac{1}{n}\right)\right\},$$

which evidently converge to the point (0, 0) we see that for all n

$$f(M_n) = f\left(\frac{1}{n}, \frac{1}{n}\right) = \frac{1}{2}, \quad \text{but} \quad f(M_n') = f\left(\frac{2}{n}, \frac{1}{n}\right) = \frac{2}{5}.$$

This implies that the above limit does not exist.

We advise the reader to prove in the same way that the limit

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2-y^2}{x^2+y^2}$$

does not exist.

(4) However, the limit

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2y}{x^2+y^2} = 0.$$

does exist. This follows at once from the inequality

$$\left|\frac{x^2y}{x^2+y^2}\right| \leqslant \frac{1}{2}|x|.$$

131. Repeated limits. Besides the limit of the function $f(x_1, \ldots, x_m)$ considered above when all arguments tend to their limits simultaneously, we encounter limits of a different kind obtained when each argument tends to its limit successively, the passages occurring in a prescribed order. The first type of limit is called an *m-tuple* (or double, triple, etc., for $m = 2, 3, \ldots$) while the latter type is called a *repeated* limit.

For simplicity we shall confine ourselves to the case of the function of two variables $f(x, y)$. Moreover, we assume that the domain $\mathcal{M}$ of variation of x, y is such that x (independently of y) can take any value in a set $\mathcal{X}$ for which a is a point of condensation, but does not belong to $\mathcal{X}$, and similarly, y (independently of x) varies over the set $\mathcal{Y}$ with the point of condensation b which does not belong to $\mathcal{Y}$. Such a domain $\mathcal{M}$ could symbolically be denoted by $\mathcal{X}\times\mathcal{Y}$. For instance,

$$(a, a+H;\ b, b+K) = (a, a+H)\times(b, b+K).$$

If for a *fixed* y in $\mathcal{Y}$ there exists for the function $f(x, y)$ (which is a function of x only) the limit as $x\to a$ this limit in general depends on the fixed y, i.e.

$$\lim_{x\to a} f(x,\ y) = \varphi(y).$$

Now we can consider the problem of the limit of the function $\varphi(y)$ as $y\to b$:

$$\lim_{y\to b} \varphi(y) = \lim_{y\to b}\lim_{x\to a} f(x, y);$$

this problem is one of repeated limits. The second limit is obtained if we carry out the operations in the reverse order:

$$\lim_{x \to a} \lim_{y \to b} f(x, y).$$

The repeated limits are not necessarily equal.

If, for instance, in the domain $\mathcal{M}$ $(0, +\infty; 0, +\infty)$ we take

$$(1) \qquad f(x, y) = \frac{x - y + x^2 + y^2}{x + y},$$

so for $a = b = 0$

$$\varphi(y) = \lim_{x \to 0} f(x, y) = y - 1, \quad \lim_{y \to 0} \varphi(y) = \lim_{y \to 0} \lim_{x \to 0} f(x, y) = -1,$$

while

$$\psi(x) = \lim_{y \to 0} f(x, y) = x + 1, \quad \lim_{x \to 0} \psi(x) = \lim_{x \to 0} \lim_{y \to 0} f(x, y) = 1.$$

It may also happen that one of the repeated limits exists while the other does not. This is, for instance, the case for the functions

$$(2) \qquad f(x, y) = \frac{x \sin \frac{1}{x} + y}{x + y} \quad \text{or} \quad (3)\ f(x, y) = x \sin \frac{1}{y};$$

in both cases the repeated limit $\lim_{y \to 0} \lim_{x \to 0} f$ exists but the repeated limit $\lim_{x \to 0} \lim_{y \to 0} f$ does not exist (in the last example even the ordinary limit $\lim_{y \to 0} f$ does not exist).

These simple examples indicate how cautious we have to be in changing the order of two limits with respect to different variables: many erroneous results may follow from such an illegitimate operation. Many important problems of analysis are connected with the changing of the order of limits and thus, evidently, each time the legitimacy of such an operation should be proved.

One case is covered by the following important theorem which simultaneously establishes the connection between the double and repeated limits.

THEOREM. *If (1) there exists the double limit (finite or otherwise)*

$$A = \lim_{\substack{x \to a \\ y \to b}} f(x, y)$$

and (2) for any y in $\mathcal{Y}$ the ordinary limit with respect to x

$$\varphi(y) = \lim_{x \to a} f(x, y)$$

exists and is finite, then the repeated limit

$$\lim_{y\to b}\varphi(y)=\lim_{y\to b}\lim_{x\to a}f(x,y)$$

exists, and is equal to the double limit.

We prove the theorem for finite A, a and b. According to the definition of the limit of a function in the "ε-δ language" [Sec. 129], for a given $\varepsilon>0$ a number $\delta>0$ can be found such that, provided,

$$|f(x,y)-A|<\varepsilon, \tag{11}$$

$|x-a|<\delta$ and $|y-b|<\delta$ (x being in $\mathfrak{X}$ and y in $\mathfrak{Y}$). We now fix y so that the inequality $|y-b|<\delta$ holds and in (11) we pass to the limit as $x\to a$. Since, by *(2)*, $f(x, y)$ tends to the limit $\varphi(y)$, we obtain

$$|\varphi(y)-A|\leqslant\varepsilon.$$

Remembering that y is an arbitrary number in $\mathfrak{Y}$ such that $|y-b|<\delta$, we find that

$$A=\lim_{y\to b}\varphi(y)=\lim_{y\to b}\lim_{x\to a}f(x,y).$$

This completes the proof.

If besides conditions *(1)* and *(2)* there exists for an arbitrary x in $\mathfrak{X}$ a (finite) ordinary limit

$$\psi(x)=\lim_{y\to b}f(x,y),$$

then it follows from the above, that if x and y be exchanged, the repeated limit

$$\lim_{x\to a}\psi(x)=\lim_{x\to a}\lim_{y\to b}f(x,y)$$

also exists, and is equal to the same number A; in this case the repeated limits are equal.

This theorem implies that in Examples *(1)* and *(2)* the double limit does not exist. This can also be verified directly.

However, in Example *(3)* the double limit exists: we observe from the inequality

$$\left|x\sin\frac{1}{y}\right|\leqslant|x|$$

that it is zero. This example shows that condition *(1)* of the theorem does not imply condition *(2)*.

However, the existence of the double limit is not *necessary* for the existence of the repeated limits; in Example *(3)* of the preceding section both repeated limits exist and vanish, while the double limit does not exist.

§ 2. Continuous functions

132. Continuity and discontinuities of functions of several variables. Suppose that the function $f(x_1, \ldots, x_m)$ is defined in a set $\mathcal{M}$ of points of an m-dimensional space and $M'(x_1', \ldots, x_m')$ is a point of condensation of the set, and belongs to the set.

We say that *a function $f(x_1, \ldots, x_m)$ is continuous at the point $M'(x_1', \ldots, x_m')$ if the relation*

$$\lim_{\substack{x_1 \to x_1' \\ \cdots\cdots \\ x_m \to x_m'}} f(x_1, \ldots, x_m) = f(x_1', \ldots, x_m') \tag{1}$$

holds. Otherwise the function has a discontinuity at the point M'.

In the "ε–δ language" the continuity of a function at the point M is formulated as follows [Sec. 129].

For an arbitrary $\varepsilon > 0$ a number $\delta > 0$ can be found such that

$$|f(x_1, \ldots, x_m) - f(x_1', \ldots, x_m')| < \varepsilon, \tag{2}$$

provided

$$|x_1 - x_1'| < \delta, \quad \ldots, \quad |x_m - x_m'| < \delta, \tag{3}$$

or in other words: for $\varepsilon > 0$ a number $r > 0$ can be found such that

$$|f(M) - f(M')| < \varepsilon,$$

provided the distance

$$\overline{MM'} < r.$$

The point M is assumed to belong to the set $\mathcal{M}$; in particular, it may coincide with M'. For this reason the limit of the function at the point M' is identical with the value of the function at this point; the usual requirement that M is distinct from M' is superfluous.

Considering the differences $x_1 - x_1', \ldots, x_m - x_m'$ as increments $\Delta x_1, \ldots, \Delta x_m$ of the independent variables and the difference

$$f(x_1, \ldots, x_m) - f(x_1', \ldots, x_m')$$

as the increment of the function, we may state that (as in the case of a function of one variable) the function is continuous if, to infinitesimal increments of the independent variables there corresponds an infinitesimal increment of the function.

When defined in the above manner, continuity of the function at the point M' is, we may say, continuity with respect to the set of variables $x_1, x_2, \ldots, x_m$. If the function is continuous in this sense, then also

$$\lim_{x_1 \to x_1'} f(x_1, x_2', \ldots, x_m') = f(x_1', x_2', \ldots, x_m'),$$

$$\lim_{\substack{x_1 \to x_1' \\ x_2 \to x_2'}} f(x_1, x_2, x_3', \ldots, x_m') = f(x_1', x_2', x_3', \ldots, x_m'),$$

etc., since we have performed here partial approximations of M to M'. In other words, the function is continuous separately with respect to each variable x_i, to each pair of variables x_i, x_j, etc.

We have already encountered examples of continuous functions. Thus, in Sec. 130 we established the continuity of the integral and fractional rational functions of m arguments at all points of the m-dimensional space (for the fractional function except at the points at which its denominator vanishes). In the same section, in *(2)* we proved the continuity of the power-exponential function x^y for all points of the right semi-plane $(x > 0)$.

If we again examine the function

$$f(x, y) = \frac{xy}{x^2 + y^2} \quad (\text{for } x^2 + y^2 > 0)$$

defined by this formula in the entire plane, except at the origin, and we set $f(0, 0) = 0$ we arrive at an example of a discontinuity. It occurs at the origin, since [Sec. 130, *(3)*] as $x \to 0$, $y \to 0$ the limit of the function does not exist.

We note the following interesting phenomenon. The function $f(x, y)$, considered in the previous paragraph, is not continuous at the point (0, 0) with respect to the set of both variables, but it is separately continuous at this point both with respect to x and to y; this result follows from the fact that $f(x, 0) = f(0, y) = 0$. Incidentally this is to be expected if we realize that when speaking of the continuity with respect to x and y separately, we consider the approach to the point (0, 0) along the x-axis only (or along the y-axis only), disregarding an infinite variety of other ways of approach.

Remark. Cauchy in his *Algebraic Analysis* attempted to prove that a function of several variables separately continuous with respect to each variable is also continuous with respect to the set of the variables. The preceding example disproves this statement.

If for the function $f(M)$ as M tends to M' there exists no definite finite limit

$$\lim_{M \to M'} f(M),$$

we say that at point M' the function has a discontinuity, even if at the point M' itself the function is not defined.

133. Operations on continuous functions. It is easy to formulate and to prove a theorem on the continuity of the sum, difference, product and quotient of two continuous functions [see Sec. 62]. We leave this to the reader.

We investigate here only the theorem on superposition of two continuous functions. Just as in Sec. 128 we assume that besides the function $u = f(x_1, \ldots, x_m)$ given over the set $\mathcal{M}$ of m-dimensional points $M(x_1, \ldots, x_m)$ we are given m functions

$$x_1 = \varphi_1(t_1, \ldots, t_k), \ \ldots, \ x_m = \varphi_m(t_1, \ldots, t_k) \tag{4}$$

over a set $\mathcal{P}$ of k-dimensional points $P(t_1, \ldots, t_k)$, the point M with coordinates (4) lying within the boundary of the set $\mathcal{M}$.

THEOREM. *If the functions $\varphi_i(P)$ $(i = 1, \ldots, m)$ are all continuous at the point $P'(t_1', \ldots, t_k')$ in $\mathcal{P}$ and the function $f(M)$ is continuous at the corresponding point $M'(x_1', \ldots, x_m')$ with the coordinates*

$$x_1' = \varphi_1(t_1', \ldots, t_k'), \ \ldots, \ x_m' = \varphi_m(t_1', \ldots, t_k'),$$

then the compound function

$$u = f\big(\varphi_1(t_1, \ldots, t_k), \ \ldots, \ \varphi_m(t_1, \ldots, t_k)\big) = f\big(\varphi_1(P), \ldots, \varphi_m(P)\big)$$

is continuous at the point P'.

Proof. First for $\varepsilon > 0$ the number $\delta > 0$ is determined, such that (3) implies (2) by the continuity of function (f). Next, for the number δ (by the continuity of the functions $\varphi_1, \ldots, \varphi_m$), a number $\eta > 0$ can be found such that the inequalities

$$|t_1 - t_1'| < \eta, \ \ldots, \ |t_k - t_k'| < \eta \tag{5}$$

imply the inequalities

$$|x_1 - x_1'| = |\varphi_1(t_1, \ldots, t_k) - \varphi_1(t_1', \ldots, t_k')| < \delta, \ \ldots$$
$$|x_m - x_m'| = |\varphi_m(t_1, \ldots, t_k) - \varphi_m(t_1', \ldots, t_k')| < \delta.$$

Then, by (5) we have

$$|f(x_1, \ldots, x_m) - f(x_1', \ldots, x_m')|$$
$$= |f\big(\varphi_1(t_1, \ldots, t_k), \ldots, \varphi_m(t_1, \ldots, t_k)\big)$$
$$- f\big(\varphi_1(t_1', \ldots, t_k'), \ldots, \varphi_m(t_1', \ldots, t_k')\big)| < \varepsilon.$$

This completes the proof.

134. Theorem on the vanishing of a function. We now proceed to examine the properties of functions of several variables, continuous at all points of a domain $\mathcal{D}$ (or, briefly, continuous in the domain $\mathcal{D}$) of an m-dimensional space†. They are analogous to the properties of a function of one variable continuous in an interval [Chapter 4, § 2].

For brevity, we shall confine ourselves to the case of two independent variables. The extension to the general case can be carried out directly and does not present any difficulties. Incidentally, some remarks will be made about this problem.

In order to formulate the theorem analogous to the first Bolzano–Cauchy theorem [Sec. 68], we require the concept of a *connected domain*: this is a domain in which any two points can be joined by a *broken line* [Sec. 125] lying wholly in the domain.

THEOREM. *Suppose that a function $f(x, y)$ is defined and is continuous in a connected domain $\mathcal{D}$. If at two points $M'(x', y')$ and $M''(x'', y'')$ of this domain the function takes values of distinct signs, i.e.*

$$f(x', y') < 0, \quad f(x'', y'') > 0,$$

then there exists in this domain a point $M_0(x_0, y_0)$ at which the function vanishes, $f(x_0, y_0) = 0$.

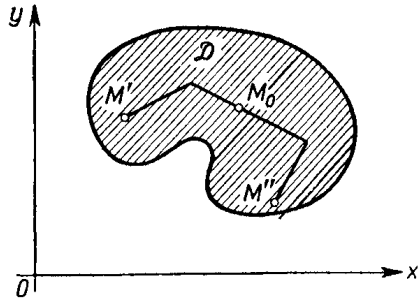

FIG. 57.

The proof will be based on reducing the problem to the case of a function of one independent variable.

By the connectedness of the domain $\mathcal{D}$, the points M' and M'' can be joined by a broken line lying in $\mathcal{D}$ (Fig. 57). If at any of the

† The word "domain" is understood in the sense of Sec. 127.

vertices the function $f(x, y)$ vanishes, then the statement of the theorem is true. Otherwise, moving along the segments of the line we necessarily arrive at a segment of a straight line on the ends of which the function takes values of distinct signs. Thus, without loss of generality we could assume from the very beginning that the segment $M'M''$ of the straight line having the equations

$$x = x' + t(x'' - x'), \quad y = y' + t(y'' - y')$$
$$(0 \leqslant t \leqslant 1),$$

wholly belongs to the domain $\mathcal{D}$. If the point $M(x, y)$ moves along this segment, the original function $f(x, y)$ becomes a compound function of the new variable t:

$$F(t) = f\big(x' + t(x'' - x'),\ y' + t(y'' - y')\big);$$

according to the theorem of the preceding section this function is continuous. Now for $F(t)$ we have

$$F(0) = f(x',\ y') < 0 \quad \text{and} \quad F(1) = f(x'',\ y'') > 0.$$

Applying to the function $F(t)$ the theorem proved in Sec. 68 we find that $F(t_0) = 0$ for a value of t_0 between zero and unity. Bearing in mind the definition of the function $F(t)$ we therefore obtain

$$f(x_0, y_0) = 0,$$

where

$$x_0 = x' + t_0(x'' - x'), \quad y_0 = y' + t_0(y'' - y').$$

The point $M_0(x_0, y_0)$ is the required one.

Hence we have deduced a theorem analogous to the second Bolzano–Cauchy theorem (incidentally, it can be deduced directly).

The reader should observe that the extension to the space of m dimensions (for $m > 2$) does not lead to any difficulties, since in an m-dimensional connected domain the points can be connected by a broken line and the problem is thus reduced, as above, to an investigation of a function of one variable.

135. The Bolzano–Weierstrass lemma. In further investigations we shall need a generalization of the lemma of Sec. 51 in the case of a sequence of points in a domain of a space of an arbitrary number of dimensions. We agree to call a set of points $\mathcal{M}$ in this space bounded if this set is contained in a parallelepiped. As before, we consider the "plane" case only.

Bolzano–Weierstrass lemma. *From an arbitrary bounded sequence of points*

$$M_1(x_1, y_1),\ M_2(x_2, y_2), \ldots,\ M_n(x_n, y_n),\ \ldots$$

we can always extract a partial sequence

$$M_{n_1}(x_{n_1}, y_{n_1}),\ M_{n_2}(x_{n_2}, y_{n_2}),\ \ldots,\ M_{n_k}(x_{n_k}, y_{n_k}),\ \ldots$$
$$(n_1 < n_2 < \ldots < n_k < \ldots,\ n_k \to +\infty)$$

which converges to a limit point.

Proof. This is most easily carried out if we make use of the lemma proved in Sec. 51 for the case of a linear sequence.

Since our sequence is assumed to be bounded, all its points are contained in a rectangle $[a, b; c, d]$. Hence

$$a \leqslant x_n \leqslant b,\ c \leqslant y_n \leqslant \partial \quad (\text{for } n = 1, 2, 3, \ldots).$$

Applying the lemma of Sec. 51 first to the sequence $\{x_n\}$, we extract a partial sequence $\{x_{n_k}\}$ converging to a limit $\bar{x}$. Thus for the partial sequence of points

$$(x_{n_1}, y_{n_1}),\quad (x_{n_2}, y_{n_2}),\ \ldots,\ (x_{n_k}, y_{n_k}),\ \ldots$$

the first coordinates already have a limit. We now apply the theorem to the sequence of the second coordinates $\{y_{n_k}\}$ and extract a partial sequence $\{y_{n_{k_i}}\}$ which also tends to a limit $\bar{y}$. It is then evident that the partial sequence of points

$$(x_{n_{k_1}}, y_{n_{k_1}}),\ (x_{n_{k_2}}, y_{n_{k_2}}),\ \ldots,\ (x_{n_{k_i}}, y_{n_{k_i}}),\ \ldots$$

tends to the limiting point $(\bar{x}, \bar{y})$.

This reasoning can also easily be extended to the case of $m > 2$ dimensions, only the extraction of the partial sequences in the general case would have to be repeated not twice but m times.

136. Theorem on the boundedness of a function. With the aid of of the above theorem we can easily establish the first Weierstrass theorem for functions of two variables.

THEOREM. *If a function $f(x, y)$ is defined and is continuous in a bounded closed domain $\mathcal{D}$†, then it is bounded above and below, i.e. all its values are contained between two finite limits*

$$m \leqslant f(x, y) \leqslant M.$$

† Now it need not be connected.

Proof. This (by assuming the converse) is entirely analogous to the reasoning of Sec. 72. Suppose that the function $f(x, y)$ when (x, y) varies over $\mathcal{D}$ is unbounded, say above. Then for any n a point $M_n(x_n, y_n)$ in $\mathcal{D}$ can be found, such that

$$f(x_n, y_n) > n. \tag{6}$$

According to the lemma of Sec. 135 we can extract from the bounded sequence $\{M_n\}$ a partial sequence $\{M_{n_k}\}$ which converges to the limit point $\overline{M}(\overline{x}, \overline{y})$.

Note that this point $\overline{M}$ must belong to $\mathcal{D}$. In fact, were this not the case, all points M_{n_k} would be distinct from it and the point $\overline{M}$ would be a point of condensation of the domain $\mathcal{D}$ not belonging to $\mathcal{D}$; this is impossible since the domain $\mathcal{D}$ is closed [Sec. 127].

Since the function is continuous at the point $\overline{M}$ we have

$$f(M_{n_k}) = f(x_{n_k}, y_{n_k}) \to f(\overline{M}) = f(\overline{x}, \overline{y}),$$

which contradicts (6).

The second Weierstrass theorem can be formulated and proved (using the preceding theorem) in exactly the same way as in Sec. 73.

Observe that without essential alterations of the reasoning, both Weierstrass theorems can be extended also to the case when the function is continuous in an arbitrary closed set $\mathcal{M}$ (which needs be a domain).

Just as in the case of functions of one variable, for a function $f(x, y)$ defined and bounded in a set $\mathcal{M}$ the difference between the exact upper and lower bounds of the values of the function in $\mathcal{M}$ is called its *oscillation* in the set. If $\mathcal{M}$ is bounded and closed (in particular if $\mathcal{M}$ is a bounded and closed domain) and the function f is continuous in it, the oscillation is simply the difference between the greatest and the smallest values.

137. Uniform continuity. We know that the continuity of a function $f(x, y)$ at a definite point (x_0, y_0) of the set $\mathcal{M}$ over which the function is defined may be expressed in the "ε–δ language" as follows: for any $\varepsilon > 0$ a number $\delta > 0$ can be found, such that the inequality

$$|f(x, y) - f(x_0, y_0)| < \varepsilon$$

is satisfied at every point (x, y) from $\mathcal{M}$, provided

$$|x - x_0| < \delta, \quad |y - y_0| < \delta.$$

Suppose now that the function $f(x, y)$ is continuous in the whole set $\mathcal{M}$; then the question arises, whether it is possible to find for a given $\varepsilon > 0$ a number $\delta > 0$ which would be suitable in the indicated sense for all points (x_0, y_0) from $\mathcal{M}$ simultaneously. If this is possible (for an arbitrary ε), then we say that the function is *uniformly continuous in* $\mathcal{M}$.

CANTOR'S THEOREM. *If a function $f(x, y)$ is continuous in a bounded closed domain $\mathcal{D}$, then it is uniformly continuous in $\mathcal{D}$.*

The proof is carried out by assuming the converse. Suppose that for a number $\varepsilon > 0$ there does not exist any number $\delta > 0$ which would be suitable for all points (x_0, y_0) of domain $\mathcal{D}$.

Take a sequence of positive numbers which converge to zero

$$\delta_1 > \delta_2 > \ldots > \delta_n > \ldots > 0, \quad \delta_n \to 0.$$

Since none of the numbers δ_n is suitable, in the indicated sense, simultaneously for all points (x_0, y_0) of the domain $\mathcal{D}$, for every δ_n a definite point (x_n, y_n) in $\mathcal{D}$ can be found at which δ_n is not suitable. This means that there exists in $\mathcal{D}$ a point (x_n', y_n'), such that

$$|x_n' - x_n| < \delta_n, \quad |y_n' - y_n| < \delta_n,$$

and

$$|f(x_n', y_n') - f(x_n, y_n)| \geqslant \varepsilon. \tag{7}$$

From the bounded sequence of points $\{(x_n, y_n)\}$, according to the Bolzano–Weierstrass lemma we extract a partial sequence $\{(x_{n_k}, y_{n_k})\}$ such that $x_{n_k} \to \bar{x}, y_{n_k} \to \bar{y}$ and the limit point $(\bar{x}, \bar{y})$ necessarily belongs to domain $\mathcal{D}$ (since the latter is closed).

Furthermore, since

$$|x_{n_k}' - x_{n_k}| < \delta_{n_k}, \quad |y_{n_k}' - y_{n_k}| < \delta_{n_k}$$

and as k increases, $n_k \to +\infty$ and $\delta_{n_k} \to 0$, we have

$$x_{n_k}' - x_{n_k} \to 0, \quad y_{n_k}' - y_{n_k} \to 0.$$

Hence also

$$x_{n_k}' \to \bar{x}, \quad y_{n_k}' \to \bar{y}.$$

By the continuity of the function $f(x, y)$ at the point $(\bar{x}, \bar{y})$ of the domain $\mathcal{D}$, we have both

$$f(x_{n_k}, y_{n_k}) \to f(\bar{x}, \bar{y}),$$

and

$$f(x'_{n_k},\ y'_{n_k}) \to f(\bar{x},\ \bar{y}),$$

whence

$$f(x_{n_k},\ y_{n_k}) - f(x'_{n_k},\ y'_{n_k}) \to 0,$$

which contradicts relation (7). This completes the proof.

To formulate the following corollary of the theorem we need the concept of a *diameter* of a point set: this is the exact upper bound of the distances between any two points in the set.

COROLLARY. *If a function $f(x, y)$ is continuous in a domain $\mathcal{D}$, then corresponding to a given $\varepsilon > 0$ a number $\delta > 0$ can be found, such that no matter how we subdivide the domain† into partial closed domains $\mathcal{D}_1, \ldots, \mathcal{D}_k$ with diameters smaller than δ, the oscillation of the function in each part separately is smaller than ε.*

It is sufficient to take δ as the number mentioned in the definition of uniform convergence. If the diameter of a partial domain $\mathcal{D}_i$ is smaller than δ, then the distance between any two points (x, y) and (x_0, y_0) in it is smaller than δ, i.e. $\sqrt{[(x-x_0)^2+(y-y_0)^2]} < \delta$. Hence we certainly have $|x-x_0| < \delta$ and $|y-y_0| < \delta$ so that $|f(x, y) - f(x_0, y_0)| < \varepsilon$. If the points are selected in such a way that $f(x, y)$ and $f(x_0, y_0)$ are the greatest and the smallest values of the function in $\mathcal{D}_i$, respectively, we arrive at the required result.

It is readily observed that the above theorem can be extended without alterations (as for the Weierstrass theorems) to the case of a function continuous in an arbitrary bounded closed set $\mathcal{M}$.

† These partial domains may have only boundary points in common.

CHAPTER 9

DIFFERENTIATION OF FUNCTIONS OF SEVERAL VARIABLES

§ 1. Derivatives and differentials of functions of several variables

138. Partial derivatives. To simplify the notation and discussion we shall confine ourselves to the case of functions of three variables; however, all the considerations below are valid for functions of an arbitrary number of variables.

Suppose that there is defined a function $u = f(x, y, z)$ over an (open) domain $\mathcal{D}$; take a point $M_0(x_0, y_0, z_0)$ in this domain. If we ascribe to y and z constant values y_0 and z_0 and we vary x, then u is a function of one variable x in the vicinity of x_0; we consider the problem of calculating its derivative at the point x_0. Let x_0 be increased by Δx; then the function acquires the increment

$$\Delta_x u = f(x_0 + \Delta x, y_0, z_0) - f(x_0, y_0, z_0),$$

which may be called its *partial increment* (with respect to x), since it is produced by a change in one variable only. By the definition of a derivative this gives rise to the limit

$$\lim_{\Delta x \to 0} \frac{\Delta_x u}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0, z_0) - f(x_0, y_0, z_0)}{\Delta x}.$$

This is called the *partial derivative* of the function $f(x, y, z)$ with respect to x at the point (x_0, y_0, z_0).

We observe that in this definition not all the coordinates are on an equal basis, since y_0 and z_0 are fixed whereas x changes, tending to x_0.

The partial derivative is denoted by one of the symbols

$$\frac{\partial u}{\partial x},\ \frac{\partial f(x_0, y_0, z_0)}{\partial x}\dagger;\quad u'_x,\ f'_x\ (x_0,\ y_0,\ z_0);\quad D_x u,\ D_x f(x_0,\ y_0,\ z_0).$$

Observe that the letter x indicates only the variable with respect to which the derivative is taken and is not connected with the point (x_0, y_0, z_0) at which the derivative is calculated‡.

Similarly, regarding x and z as constants and y as variable we may consider the limit

$$\lim_{\Delta y \to 0} \frac{\Delta_y u}{\Delta y} = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y, z_0) - f(x_0, y_0, z_0)}{\Delta y}.$$

This limit is called the partial derivative of the function $f(x, y, z)$ with respect to y at the point (x_0, y_0, z_0), and is denoted by means of the symbols

$$\frac{\partial u}{\partial y},\quad \frac{\partial f(x_0, y_0, z_0)}{\partial y};\quad u'_y, f'_y, (x_0, y_0, z_0);\quad D_y u,\ D_y f(x_0, y_0, z_0).$$

In exactly the same way the partial derivative with respect to of the function $f(x, y, z)$ at the point (x_0, y_0, z_0) is defined.

The actual computation of the partial derivative is essentially the same operation as in the case of an ordinary derivative.

Examples. (1) Set $u = x^y\ (x > 0)$; the partial derivatives of this function are the following:

$$\frac{\partial u}{\partial x} = yx^{y-1},\quad \frac{\partial u}{\partial y} = x^y \cdot \log x.$$

The first is calculated as the derivative of the power function of x (for $y = \text{const}$), the second as the derivative of the exponential function of y (for $x = \text{const}$).

(2) If $u = \arctan (x/y)$ we have

$$\frac{\partial u}{\partial x} = \frac{y}{x^2 + y^2},\quad \frac{\partial u}{\partial y} = -\frac{x}{x^2 + y^2}.$$

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† We employ the "round" partial differential ∂ (instead of an ordinary d) in denoting the partial derivative.

‡ Similarly, we use the symbols

$$\frac{\partial f}{\partial x},\ f'_x,\ D_x f$$

to denote the partial derivative of the function $f(x, y, z)$ with respect to x. Such remarks will in future be omitted.

(3) For $u = x/(x^2+y^2+z^2)$ we obtain

$$\frac{\partial u}{\partial x} = \frac{y^2+z^2-x^2}{(x^2+y^2+z^2)^2}, \quad \frac{\partial u}{\partial y} = -\frac{2xy}{(x^2+y^2+z^2)^2}, \quad \frac{\partial u}{\partial z} = -\frac{2xz}{(x^2+y^2+z^2)^2},$$

It should be observed that the above symbols for partial derivatives (with the "round" ∂) must be regarded as entire symbols, and not as quotients or fractions.

139. Total increment of the function. Consider increments Δx, Δy, Δz of the three independent variables at $x = x_0$, $y = y_0$, $z = z_0$; then the function $u = f(x, y, z)$ has an increment

$$\begin{aligned}\Delta u &= \Delta f(x_0, y_0, z_0)\\ &= f(x_0+\Delta x, y_0+\Delta y, z_0+\Delta z) - f(x_0, y_0, z_0),\end{aligned}$$

which is called the *total* increment of the function.

In the case of the function $y = f(x)$ of one variable, assuming the existence at the point x_0 of a (finite) derivative $f'(x_0)$, the increment of the function is given by the formula [Sec. 82, (2)]

$$\Delta y = \Delta f(x_0) = f'(x_0)\Delta x + \alpha \Delta x,$$

where α depends on Δx and tends to zero as $\Delta x \to 0$.

We propose to establish an analogous formula for increments of the function $u = f(x, y, z)$:

$$\begin{aligned}\Delta u &= \Delta f(x_0, y_0, z_0)\\ &= f'_x(x_0, y_0, z_0)\Delta x + f'_y(x_0, y_0, z_0)\Delta y + f'_z(x_0, y_0, z_0)\Delta z\\ &\qquad + \alpha \Delta x + \beta \Delta y + \gamma \Delta z \quad (1)\end{aligned}$$

where α, β, γ depend on $\Delta x, \Delta y, \Delta z$ and tend to zero, as do the latter. However, we shall now have to impose more severe restrictions on the function.

(1). If the partial derivatives $f'_x(x, y, z), f'_y(x, y, z), f'_z(x, y, z)$ exist not only at the point (x_0, y_0, z_0) but also in a neighbourhood of this point, and, moreover, if they are continuous (as functions of x, y, z) at this point, then formula (1) holds.

To prove this assertion we represent the total increment of the function Δu in the form

$$\begin{aligned}\Delta u = {} & [f(x_0+\Delta x, y_0+\Delta y, z_0+\Delta z) - f(x_0, y_0+\Delta y, z_0+\Delta z)]\\ & + [f(x_0, y_0+\Delta y, z_0+\Delta z) - f(x_0, y_0, z_0+\Delta z)]\\ & + [f(x_0, y_0, z_0+\Delta z) - f(x_0, y_0, z_0)].\end{aligned}$$

Each of the above differences constitutes a partial increment of the function with respect to one variable only. Since we have assumed the existence of the partial derivatives in the vicinity of the point (x_0, y_0, z_0) for sufficiently small $\Delta x, \Delta y, \Delta z$ we may apply the formula of finite increments [Sec. 102]† to each of these differences separately; thus we obtain

$$\Delta u = f'_x(x_0 + \theta\Delta x,\ y_0 + \Delta y,\ z_0 + \Delta z)\,\Delta x + f'_y(x_0,\ y_0 + \theta_1\Delta y,\ z_0 + \Delta z)\,\Delta y + f'_z(x_0,\ y_0,\ z_0 + \theta_2\Delta z)\,\Delta z.$$

Setting here

$$f'_x(x_0 + \theta\Delta x,\ y_0 + \Delta y,\ z_0 + \Delta z) = f'_x(x_0,\ y_0,\ z_0) + \alpha,$$
$$f'_y(x_0,\ y_0 + \theta_1\Delta y,\ z_0 + \Delta z) = f'_y(x_0,\ y_0,\ z_0) + \beta,$$
$$f'_z(x_0,\ y_0,\ z_0 + \theta_2\Delta z) = f'_z(x_0,\ y_0,\ z_0) + \gamma,$$

we arrive at expression (1) for Δu. As $\Delta x \to 0, \Delta y \to 0, \Delta z \to 0$ the arguments of the derivatives on the left-hand sides of these relations tend to x_0, y_0, z_0 (for $\theta, \theta_1, \theta_2$ are regular fractions); consequently, the derivatives themselves, by the assumptions about the continuity of the variables for these values, tend to the derivatives on the right-hand sides, while the quantities α, β, γ tend to zero. This completes the proof.

Incidentally, the above theorem makes it possible to establish the following assertion:

(2) The existence and continuity of partial derivatives at a given point imply the continuity of the function itself.

In fact, if $\Delta x \to 0, \Delta y \to 0, \Delta z \to 0$, then evidently also $\Delta u \to 0$.

To write formula (1) in a more compact form we introduce the expression

$$\varrho = \sqrt{(\Delta x^2 + \Delta y^2 + \Delta z^2)},$$

i.e. the distance between the points (x_0, y_0, z_0) and $(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z)$.

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† Taking, for instance, the first difference, it may be regarded as the increment of the function $f(x, y_0 + \Delta y, z_0 + \Delta z)$ of one variable x, corresponding to the passage from $x = x_0$ to $x = x_0 + \Delta x$. The derivative with respect to x of this function, i.e. $f'_x(x, y_0 + \Delta y, z_0 + \Delta z)$, in accordance with the assumption, exists for all values of x in the interval $[x_0, x_0 + \Delta x]$, and therefore the formula of finite differences is applicable.

Now we may write

$$\alpha\Delta x+\beta\Delta y+\gamma\Delta z=\left(\alpha\frac{\Delta x}{\varrho}+\beta\frac{\Delta y}{\varrho}+\gamma\frac{\Delta z}{\varrho}\right)\varrho.$$

Denoting the expression in parenthesis by ε we obtain

$$\alpha\Delta x+\beta\Delta y+\gamma\Delta z=\varepsilon\varrho,$$

where ε depends on $\Delta x, \Delta y, \Delta z$, and it tends to zero as $\Delta x\to 0$, $\Delta y\to 0, \Delta z\to 0$, or, briefly, as $\varrho\to 0$. Thus formula (1) can be now written in the form

$$\Delta u=\Delta f(x_0, y_0, z_0)=f'_x(x_0, y_0, z_0)\,\Delta x+f'_y(x_0, y_0, z_0)\,\Delta y+f'_z(x_0, y_0, z_0)\,\Delta z+\varepsilon\varrho, \quad (2)$$

where $\varepsilon\to 0$ as $\varrho\to 0$. It is evident that the quantity $\varepsilon\varrho$ may be written $o(\varrho)$ (if we extend the notation introduced in Sec. 54 to the case of functions of several variables).

Observe that in our argument we have not formally excluded the case in which the increments $\Delta x, \Delta y, \Delta z$ vanish separately or simultaneously. Thus, when speaking of the limit relations

$$\alpha\to 0, \quad \beta\to 0, \quad \gamma\to 0, \quad \varepsilon\to 0$$

for $\Delta x\to 0, \Delta y\to 0, \Delta z\to 0$, we understood them in the wider sense, i.e. without excluding the possibility that these increments may vanish in the course of variation. [*See* an analogous remark in Sec. 82.]

In proving the preceding theorem we imposed on the function of several variables more severe restrictions than on a function of one variable. To prove that these conditions are necessary for the validity (1) or (2), we consider the following example (dealing, for simplicity, with a function of two variables only).

We define the function $f(x, y)$ by the relations

$$f(x, y)=\frac{x^2y}{x^2+y^2} \quad (\text{if} \quad x^2+y^2>0), \quad f(0, 0)=0.$$

This function is continuous over the entire plane; for the point (0, 0) continuity follows from Sec.130, *(4)*. Furthermore, the partial derivatives with respect to x and y also exist over the entire plane. It is evident that for $x^2+y^2>0$ we have

$$f'_x(x, y)=\frac{2xy^3}{(x^2+y^2)^2}, \quad f'_y(x, y)=\frac{x^2(x^2-y^2)}{(x^2+y^2)^2}.$$

At the origin we have $f'_x(0, 0) = f'_y(0, 0) = 0$; this result follows directly from the definition of partial derivatives and from the fact that $f(x, 0) = f(0, y) = 0$. It can easily be proved that the derivatives are discontinuous at the point (0, 0) (for example, set $y = x = 1/n \to 0$).

A formula of the form (1) or (2) does not occur for our function at the point (0, 0). In fact, assuming the converse, we would have

$$\Delta f(0, 0) = \frac{\Delta x^2 \Delta y}{\Delta x^2 + \Delta y^2} = \varepsilon \sqrt{(\Delta x^2 + \Delta y^2)},$$

where $\varepsilon \to 0$ as $\Delta x \to 0, \Delta y \to 0$. Setting in particular $\Delta y = \Delta x > 0$ we have

$$\frac{1}{2}\Delta x = \varepsilon \sqrt{2} \cdot \Delta x \quad \text{whence} \quad \varepsilon = \frac{1}{2\sqrt{2}}$$

and ε does not tend to zero as $\Delta x \to 0$, which contradicts the assumption.

140. Derivatives of compound functions. As an example of application of the derived formula (1) consider the problem of differentiation of compound functions. Suppose that the function

$$u = f(x, y, z)$$

is defined in a domain $\mathcal{D}$ and each of the variables x, y, z is a function of the variable t in an interval, i.e.

$$x = \varphi(t), \quad y = \psi(t), \quad z = \chi(t).$$

Assume moreover that when t varies the point (x, y, z) does not leave the domain $\mathcal{D}$.

Substituting the values of x, y and z into the function u we obtain the compound function

$$u = f(\varphi(t), \psi(t), \chi(t)).$$

Assume that u has continuous partial derivatives u'_x, u'_y, u'_z with respect to x, y, z, and that x'_t, y'_t, z'_t exist. Then we can prove the existence of the derivative of the compound function and can calculate it.

In fact, consider an increment Δt of the variable t; then Δx, $\Delta y, \Delta z$ are the corresponding increments of x, y, z, and Δu is that of the function u.

Representing the increment of u in the form (1) (this we can do since we have assumed the existence of continuous partial derivatives u'_x, u'_y, u'_z), we obtain

$$\Delta u = u'_x \Delta x + u'_y \Delta y + u'_z \Delta z + \alpha \Delta x + \beta \Delta y + \gamma \Delta z,$$

where $\alpha, \beta, \gamma \to 0$ as $\Delta x, \Delta y, \Delta z \to 0$. Dividing by Δt we have,

$$\frac{\Delta u}{\Delta t} = u'_x \frac{\Delta x}{\Delta t} + u'_y \frac{\Delta y}{\Delta t} + u'_z \frac{\Delta z}{\Delta t} + \alpha \frac{\Delta x}{\Delta t} + \beta \frac{\Delta y}{\Delta t} + \gamma \frac{\Delta z}{\Delta t}.$$

Suppose now that the increment Δt tends to zero; then $\Delta x, \Delta y, \Delta z$ tend to zero, since the functions x, y, z of t are continuous (we assumed the existence of the derivatives x'_t, y'_t, z'_t) and therefore α, β, γ also tend to zero. In the limit we obtain

$$u'_t = u'_x x'_t + u'_y y'_t + u'_z z'_t. \tag{3}$$

We observe that under the above assumptions the derivative of a compound function does exist. Making use of the differential notation we may rewrite formula (3) in the form

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}. \tag{4}$$

We examine now the case when x, y, z depend not on one variable but on several variables, for instance

$$x = \varphi(t, v), \quad y = \psi(t, v), \quad z = \chi(t, v),$$

Besides the existence and continuity of the partial derivatives of the function $f(x, y, z)$ we assume here the existence of the derivatives of the functions x, y, z with respect to t and v.

After substituting the functions φ, ψ and χ into the function f we arrive at a function of two variables t and v. Now the problem arises of the existence and calculation of the partial derivatives u'_t and u'_v. This case, however, is not essentially different from that investigated above, since in computing the partial derivative of a function of two variables we fix one of them and we are left with a function of one variable only. Consequently formula (3) for this case is unaltered and formula (4) can be rewritten in the form

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}. \tag{4a}$$

141. Examples. *(1)* Consider the power-exponential function

$$u = x^y.$$

Setting $x = \varphi(t)$, $y = \psi(t)$ and differentiating in accordance with the above rule for a compound function, we arrive at the familiar formula of Leibniz and J. Bernoulli

$$u'_t = y\,x^{y-1}\,x'_t + x^y \log x\, y'_t.$$

We have already established (in a different notation) this formula by means of an artificial device [Sec. 85, *(5)*].

Formula (3) resembles the formula $u'_t = u'_x x'_t$ for the function u of one variable x. However, we emphasize that there is a difference in the conditions under which the two formulae are derived. If u depends on one variable it is sufficient to assume the existence of the derivative u'_x, while in the case of several variables we have to assume moreover the continuity of the derivatives u'_x, u'_y, ... The following example indicates that the mere existence of these derivatives is, in general, insufficient to ensure the validity of formula (3).

(2) Define the function $u = f(x, y)$ setting

$$f(x, y) = \frac{x^2 y}{x^2 + y^2} \quad (\text{for } x^2 + y^2 > 0), \quad f(0, 0) = 0.$$

We know that these functions have partial derivatives at all points including (0, 0), and

$$f'_x(0, 0) = 0, \quad f'_y(0, 0) = 0.$$

Observe that at this point the derivatives possess a discontinuity.

Introducing a new variable t by setting $x = y = t$ we arrive at a compound function of t. According to formula (3) the derivative of this function for $t = 0$ would be equal to

$$u'_t = u'_x x'_t + u'_y y'_t = 0.$$

On the other hand, however, if we in fact substitute the values of x and y into the given function $u = f(x, y)$ we obtain for $t \neq 0$

$$u = \frac{t^2 t}{t^2 + t^2} = \frac{1}{2} t,$$

which is valid for $t = 0$ as well.

Differentiating now directly with respect to t we have $u'_t = 1/2$ for any value of t and therefore for $t = 0$.

It turns out that formula (3) is, in this case, inapplicable.

(3) The equation $x^2/a^2 + y^2/b^2 = 1$ defines the variable y as a function of x:

$$y = \pm \frac{b}{a} \sqrt{(a^2 - x^2)} \quad (-a < x < a),$$

which has the derivative

$$y'_x = \mp \frac{b}{a} \frac{x}{\sqrt{(a^2 - x^2)}} = -\frac{b^2}{a^2} \frac{x}{y}.$$

It is required to find this derivative *without solving the equation with respect to y.*

Solution. Imagine that the above function is substituted into the equation, thus replacing y; then the equation is satisfied identically. Differentiating this identity with respect to x (making use of the rule of differentiation of a compound function) we obtain

$$\frac{2x}{a^2}+\frac{2y}{b^2}y'_x=0,$$

whence, as before,

$$y'=-\frac{b^2}{a^2}\frac{x}{y}.$$

(4) Consider the general equation

$$F(x,\ y)=0,$$

which is insoluble with respect to y (F is continuous with its derivatives). Under known conditions [see Chapter 19 of Volume II] we can state that this equation determines y as a function of x and moreover it has a derivative (although we may not know the analytic expression of this function). In this case y is called an implicit function of x. It is required to find the derivative of the implicit function.

Solution. As in the particular case, imagine that y is replaced by the implicit function. Differentiating with respect to x the identity so obtained we have

$$F'_x(x,\ y)+F'_y(x,\ y)\,y'_x=0,$$

whence (provided $F'_y\neq 0$)

$$y'_x=-\frac{F'_x(x,\ y)}{F'_y(x,\ y)}.$$

142. The total differential. For the case of a function of one variable $y=f(x)$, we investigated in Sec. 89 the problem of representing its increment $\Delta y=\Delta f(x_0)=f(x_0+\Delta x)-f(x_0)$ in the form

$$\Delta f(x_0)=A\,\Delta x+o(\Delta x)\qquad (A=\text{const}). \tag{5}$$

It was shown [Sec. 90] that for such a representation to hold it is necessary and sufficient that, at the point $x=x_0$, there exists a finite derivative $f'(x_0)$ and the above written relation then holds, with $A=f'(x_0)$. The linear part

$$A\,\Delta x=f'(x_0)\Delta x=y'_x\,\Delta x$$

of the increment of the function was called its *differential dy.*

Proceeding to the case of a function of several variables, for instance the function $f(x, y, z)$ of three variables, defined in a (say,

open) domain $\mathcal{D}$ it is natural to consider an analogous problem of the possibility of representing the increment

$$\Delta u = \Delta f(x_0, y_0, z_0) = f(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) - f(x_0, y_0, z_0)$$

in the form

$$\Delta f(x_0, y_0, z_0) = A\,\Delta x + B\,\Delta y + C\,\Delta z + o(\varrho), \tag{6}$$

where A, B and C are constants and $\varrho = \sqrt{(\Delta x^2 + \Delta y^2 + \Delta z^2)}$.

As in Sec. 90 it is easy to prove that if the representation (6) is valid, then there exist partial derivatives with respect to each variable at the point (x_0, y_0, z_0), and

$$f'_x(x_0, y_0, z_0) = A, \quad f'_y(x_0, y_0, z_0) = B, \quad f'_z(x_0, y_0, z_0) = C.$$

In fact, setting for instance in (6) $\Delta y = \Delta z = 0$ and $\Delta x \neq 0$ we obtain [Sec. 90, (1a)]

$$\Delta_x f(x_0, y_0, z_0) = f(x_0 + \Delta x, y_0, z_0) - f(x_0, y_0, z_0) = A\,\Delta x + o(|\Delta x|),$$

It follows therefore that there exists

$$f'_x(x_0, y_0, z_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0, z_0) - f(x_0, y_0, z_0)}{\Delta x} = A.$$

Thus, the relation (6) can exist only in the form

$$\Delta f(x_0, y_0, z_0) = f'_x(x_0, y_0, z_0)\Delta x + f'_y(x_0, y_0, z_0)\Delta y + f'_z(x_0, y_0, z_0)\Delta z + o(\varrho), \tag{7}$$

or, briefly,

$$\Delta u = u'_x \Delta x + u'_y \Delta y + u'_z \Delta z + o(\varrho). \tag{7a}$$

However, while in the case of a function of one variable the existence of the derivative $y'_x = f'(x_0)$ at the point x_0 was sufficient for the validity of relation (5), in our case the existence of the partial derivatives

$$u'_x = f'_x(x_0, y_0, z_0), \quad u'_y = f'_y(x_0, y_0, z_0), \quad u'_z = f'_z(x_0, y_0, z_0)$$

does not ensure the validity of (6). For the case of a function of two variables this is illustrated in an example of Sec. 139. We also gave there sufficient conditions for the validity of relation (6), i.e. the existence of the partial derivatives in the vicinity of point (x_0, y_0, z_0), and their continuity at this point.

If formula (7) is valid the function $f(x, y, z)$ is said to be differentiable at the point (x_0, y_0, z_0) and (only in this case) the expression

$$u'_x \Delta x + u'_y \Delta y + u'_z \Delta z$$
$$= f'_x(x_0, y_0, z_0)\Delta x + f'_y(x_0, y_0, z_0)\Delta y + f'_z(x_0, y_0, z_0)\Delta z,$$

i.e. the linear part of the increment of the function, is called its (total) differential and is denoted by the symbol du or $df(x_0, y_0, z_0)$.

Therefore, in the case of a function of several variables, the statement that "the function is differentiable" at a point is no longer equivalent to the statement that "the function has partial derivatives with respect to all variables" at that point; the former statement means more than the latter. Incidentally we shall usually assume existence and continuity of the partial derivatives, which does ensure differentiability of the function.

We agree to call the differentials of the independent variables dx, dy, dz arbitrary increments $\Delta x, \Delta y, \Delta z$†. Hence we may write

$$df(x_0, y_0, z_0) = f'_x(x_0, y_0, z_0)dx + f'_y(x_0, y_0, z_0)dy + f'_z(x_0, y_0, z_0)dz$$

or

$$du = u'_x dx + u'_y dy + u'_z dz.$$

143. Invariance of the form of the (first) differential. Suppose that the function $u = f(x, y, z)$ has continuous partial derivatives with respect to x, y, z: u'_x, u'_y, u'_z, and x, y, z are functions of the new variables t and v, i.e.

$$x = \varphi(t, v), \quad y = \psi(t, v), \quad z = \chi(t, v),$$

which also have continuous partial derivatives $x'_t, y'_t, z'_t, x'_v, y'_v, z'_v$. Then [Sec. 140], not only do the derivatives of the compound function u with respect to t and v exist, but they are also continuous with respect to t and v. This is readily seen from (3).

---

† If we identify the differential of an independent variable x with the differential of x as a function of the independent variables x, y, z, then according to the general formula we have

$$dx = x'_x \Delta x + x'_y \Delta y + x'_z \Delta z = 1\Delta x + 0\Delta y + 0\Delta z = \Delta x.$$

Then the relation $dx = \Delta x$ is proved.

If x, y, z were independent variables, we know that the total differential of the function u would be

$$du = u'_x dx + u'_y dy + u'_z dz.$$

In our case u depends, through x, y, z, on the variables t and v. Consequently, with respect to these variables the differential has the form

$$du = u'_t dt + u'_v dv.$$

Now by virtue of (3)

$$u'_t = u'_x x'_t + u'_y y'_t + u'_z z'_t,$$

and similarly

$$u'_v = u'_x x'_v + u'_y y'_v + u'_z z'_v.$$

Substituting these values into the expression for du we have

$$du = (u'_x x'_t + u'_y y'_t + u'_z z'_t)\,dt + (u'_x x'_v + u'_y y'_v + u'_z z'_v)\,dv.$$

We group the terms as follows:

$$du = u'_x(x'_t\,dt + x'_v\,dv) + u'_y(y'_t\,dt + y'_v\,dv) + u'_z(z'_t\,dt + z'_v\,dv).$$

It is readily observed that the expressions in parenthesis are exactly the differentials of the functions x, y, z with respect to t and v. Hence we can write

$$du = u'_x dx + u'_y dy + u'_z dz.$$

We have arrived at the same form of the differential as in the case when x, y, z were independent variables (but of course the meaning of the symbols dx, dy, dz is now different).

Thus, *the (first) differential of a function of several variables has an invariant form*, just as for the case of a function of one variable†.

It may happen that x, y, z depend on different variables, for instance

$$x = \varphi(t), \quad y = \psi(t, v), \quad z = \chi(v, w).$$

In this case we may always assume that

$$x = \varphi_1(t, v, w), \quad y = \psi_1(t, v, w), \quad z = \chi_1(t, v, w),$$

and all of the previous reasoning applies to this case.

† We note that this is also true assuming only the differentiability of all the functions considered. To establish this it is sufficient to prove that the result of superposition of differentiable functions is also a differentiable function.

COROLLARIES. In the case when x and y were functions of one variable we had

$$d(cx) = c\,dx, \quad d(x \pm y) = dx \pm dy, \quad d(xy) = y\,dx + x\,dy,$$
$$d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}.$$

These formulae are also valid in the case when x, y are functions of an arbitrary number of variables, i.e. when

$$x = \varphi(t, v, \ldots), \quad y = \psi(t, v, \ldots).$$

As an example we prove the last formula.

For this purpose we first regard x and y as independent variables. Then

$$d\left(\frac{x}{y}\right) = \frac{1}{y}\,dx - \frac{x}{y^2}\,dy = \frac{y\,dx - x\,dy}{y^2}.$$

We observe that under this assumption the differential has the same form as for a function of one variable. On the basis of the invariance of the form of the differential, we can state that this formula is also valid in the case when x and y are functions of an arbitrary number of variables.

The above property of the total differential and its consequences make it possible to simplify the calculation of differentials, for instance

$$d \arctan \frac{x}{y} = \frac{1}{1 + \left(\frac{x}{y}\right)^2}\, d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{x^2 + y^2}.$$

Since the coefficients of the differentials of the independent variables are the corresponding partial derivatives, we at once obtain their values. For instance, when $u = \arctan x/y$ we have directly

$$\frac{\partial u}{\partial x} = \frac{y}{x^2 + y^2}, \quad \frac{\partial u}{\partial y} = -\frac{x}{x^2 + y^2}$$

[cf. Sec. 138, *(2)*].

144. Application of the total differential to approximate calculations. Just as for the differential of a function of one variable [Sec. 94], the total differential of a function of several variables can be used to estimate the error in approximate calculations. Suppose, for instance, that we have a function $u = f(x, y)$ and in

determining the values of x and y we make an error, say Δx and Δy. Then the value of u as well, calculated in accordance with the inaccurate values of the arguments, has an error $\Delta u = f(x+\Delta x, y+\Delta y) - f(x, y)$. We intend to estimate this error if estimates of the errors Δx and Δy are known.

Replacing (approximately) the increment of the function by its differential (which is permissible for sufficiently small values of Δx and Δy) we obtain

$$\Delta u = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y. \tag{8}$$

Here the errors Δx, Δy and the coefficients may be both positive or negative; replacing them by their absolute values we arrive at the inequality

$$|\Delta u| \leqslant \left|\frac{\partial u}{\partial x}\right| |\Delta x| + \left|\frac{\partial u}{\partial y}\right| |\Delta y|.$$

Denoting by δx, δy, δu the maximum absolute errors (or bounds of the absolute errors) we may evidently set

$$\delta u = \left|\frac{\partial u}{\partial x}\right| \delta x + \left|\frac{\partial u}{\partial y}\right| \delta y. \tag{9}$$

We now give some examples.

(1) First, with the aid of the derived formula it is easy to establish the basic rules for the use of approximate calculations. Suppose that $u = xy$ (where $x > 0$, $y > 0$) and hence $du = y\,dx + x\,dy$; replacing the differentials by the increments we obtain $\Delta u = y\Delta x + x\Delta y$ (see (8)), or passing to the bounds of the errors

$$\delta u = y\,\delta x + x\,\delta y.$$

Dividing by $u = xy$ we arrive at the final formula

$$\frac{\delta u}{u} = \frac{\delta x}{x} + \frac{\delta y}{y}, \tag{10}$$

representing the following rule: the (maximum) relative error of a product is equal to the sum of the (maximum) relative errors of the factors.

We could proceed in a simpler way, viz. first finding the logarithms in the formula $u = xy$ and then differentiating

$$\log u = \log x + \log y, \quad \frac{du}{u} = \frac{dx}{x} + \frac{dy}{y}, \quad \text{etc.}^{\dagger}$$

If $u = x/y$ we obtain in the same way

$$\log u = \log x - \log y, \quad \frac{du}{u} = \frac{dx}{x} - \frac{dy}{y};$$

---

† We draw the reader's attention to the fact that the differential of $\log u$ is calculated as if u were the independent variable, although in fact it is a function of x and y. This remark should henceforth be borne in mind.

passing to the absolute quantities and the maximum errors we arrive again at formula (10). Thus the (maximum) relative error of a quotient is equal to the sum of the (maximum) relative errors of the divisor and the divisible.

(2) One of the particular applications of the calculus of errors is in topography, mainly in calculating elements of a triangle which are not measured directly, in terms of the measured elements. We shall present an example from this field.

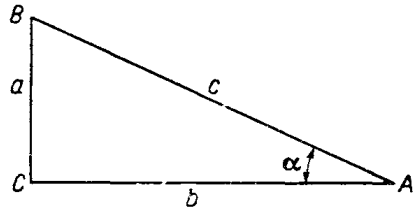

FIG. 58.

Suppose that in a right-angled triangle ABC the side $AC = b$ and the adjacent angle $BAC = \alpha$ are measured; the other side a is calculated by means of the formula $a = b \tan \alpha$. What is the influence on a of errors in measuring b and α?

Differentiating we have

$$da = \tan\alpha\, db + \frac{b}{\cos^2\alpha}\, d\alpha,$$

and hence

$$\delta a = \tan\alpha\, \delta b + \frac{b}{\cos^2\alpha}\, \delta\alpha.$$

145. Homogeneous functions. By homogeneous polynomials we mean polynomials consisting of terms all of the same degree. For instance, the expression

$$3x^2 - 2xy + 5y^2$$

is a homogeneous polynomial of degree two. Multiplying x and y by a factor t we find that the whole polynomial acquires the factor t to the power two. A similar property is true for any homogeneous polynomial.

Now, functions of a more complicated nature can also have such a property; for instance, the expression

$$x \frac{\sqrt{(x^4 + y^4)}}{x - y} \log \frac{x}{y},$$

which acquires the factor t^2 when both arguments x and y are multiplied by t; in this respect, therefore, the above expression is similar to the polynomial of the second degree. Such a function is naturally called *a homogeneous function of the second degree.*

We now give a general definition of such functions.

A function $f(x_1, \ldots, x_m)$ of m arguments defined in a domain $\mathcal{D}$ is called a *homogeneous function* of the kth degree if, on multiplying all its arguments by a factor t, the function acquires the same factor to the kth degree, i.e. if the relation

$$f(tx_1, \ldots, tx_m) = t^k f(x_1, \ldots, x_m) \tag{11}$$

is identically satisfied.

For simplicity we confine ourselves to the assumption that $x_1, \ldots, x_m$ and t take positive values only. The domain $\mathcal{D}$ over which the function f is defined is assumed to contain, together with any point $M(x_1, \ldots, x_m)$, all points of the form $M_t(tx_1, \ldots, tx_m)$ for $t > 0$, i.e. the whole ray from the origin and passing through the point M.

The degree of homogeneity k may be any real number; for instance, the function

$$x^\pi \sin\frac{y}{x} + y^\pi \cos\frac{y}{x}$$

is a homogeneous function of degree π in the arguments x and y.

We shall now attempt to derive the general expression of a homogeneous function of degree k.

First suppose that $f(x_1, \ldots, x_m)$ is a homogeneous function of zero degree; then

$$f(tx_1, tx_2, \ldots, tx_m) = f(x_1, x_2, \ldots, x_m).$$

Setting $t = 1/x$, we obtain

$$f(x_1, x_2, \ldots, x_m) = f\left(1, \frac{x_2}{x_1}, \ldots, \frac{x_m}{x_1}\right).$$

Introducing the function of $m-1$ arguments

$$\varphi(u_1, \ldots, u_{m-1}) = f(1, u_1, \ldots, u_{m-1}),$$

we find that

$$f(x_1, x_2, \ldots, x_m) = \varphi\left(\frac{x_2}{x_1}, \ldots, \frac{x_m}{x_1}\right).$$

Thus every homogeneous function of zero degree can be represented in the form of a function of the ratios of all but one of the arguments to the remaining one. Evidently the converse is also true, and therefore the preceding relation yields the general expression of a homogeneous function of zero degree.

If $f(x_1, x_2, \ldots, x_m)$ is a homogeneous function of kth degree its ratio to x_1^k is a homogeneous function of zero degree; hence

$$\frac{f(x_1, x_2, \ldots, x_m)}{x_1^k} = \varphi\left(\frac{x_2}{x_1}, \ldots, \frac{x_m}{x_1}\right),$$

and

$$f(x_1, x_2, \ldots, x_m) = x_1^k \varphi\left(\frac{x_2}{x_1}, \ldots, \frac{x_m}{x_1}\right).$$

If, on the contrary, such a relation is satisfied for a function $f(x_1, x_2, \ldots, x_m)$, then it is easy to verify that it is a homogeneous function of degree k. Thus we have arrived at the general form of a homogeneous function of degree k.

Example.

$$x \frac{\sqrt{(x^4+y^4)}}{x-y} \log \frac{x}{y} = x^2 \frac{\sqrt{\left[1+\left(\frac{y}{x}\right)^4\right]}}{\frac{y}{x}-1} \log \frac{y}{x}.$$

Assume now that a homogeneous function $f(x, y, z)$† (of degree $k = 3$) has over an (open) domain $\mathcal{D}$ continuous partial derivatives with respect to all arguments. Taking an arbitrary point (x_0, y_0, z_0) of $\mathcal{D}$ we have, by the basic identity (11), for any $t > 0$ the formula

$$f(tx_0, ty_0, tz_0) = t^k f(x_0, y_0, z_0).$$

Differentiating this relation with respect to t—the left-hand side in accordance with the rule of differentiation of a compound

† For the purpose of simplifying the formula we confine ourselves to the case of three variables.

function† and the right-hand side simply as a power function, we obtain

$$f'_x(tx_0, ty_0, tz_0)\,x_0 + f'_y(tx_0, ty_0, tz_0)\,y_0 + f'_z(tx_0, ty_0, tz_0)\,z_0 = kt^{k-1} f(x_0, y_0, z_0).$$

Setting here $t = 1$ we have

$$f'_x(x_0, y_0, z_0)\,x_0 + f'_y(x_0, y_0, z_0)\,y_0 + f'_z(x_0, y_0, z_0)\,z_0 = kf(x_0, y_0, z_0).$$

Thus for an arbitrary point (x, y, z) we have the relation

$$f'_x(x, y, z)\,x + f'_y(x, y, z)\,y + f'_z(x, y, z)\,z = kf(x, y, z), \tag{12}$$

which is called *Euler's formula.*

We know that this relation is satisfied by any homogeneous function of degree k, which has partial continuous derivatives. It can be proved that, conversely, every function which, together with its partial derivatives, is continuous and which satisfies Euler's formula, is necessarily a homogeneous function of degree k.

Remark. Euler in his *Differential Calculus* considers only particular types of homogeneous expressions — integral, rational, irrational, and their combinations — but does not give a general consideration. In deriving the formula bearing his name, however, he bases his discussion on the concept of a homogeneous function in the form of a power of one of its arguments multiplied by a function of ratios of the remaining arguments.

§ 2. Derivatives and differentials of higher orders

146. Derivatives of higher orders. If the function $u = f(x, y, z)$‡ has in an (open) domain $\mathcal{D}$ a partial derivative with respect to one of the variables, then this derivative is itself a function of x, y, z and can have at a point (x_0, y_0, z_0) partial derivatives with respect to the same or other variables. The latter derivatives are partial derivatives of the second order (or second partial derivatives) of the original function.

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† It is permissible to apply this rule since we have assumed continuity of the partial derivatives [Sec. 140].

‡ For simplicity we again confine ourselves to the case of three variables.

If the first derivative were taken with respect to x, say, its derivatives with respect to x, y, z are denoted by the symbols

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 f(x_0, y_0, z_0)}{\partial x^2}, \quad \frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial^2 f(x_0, y_0, z_0)}{\partial x\, \partial y},$$

$$\frac{\partial^2 u}{\partial x\, \partial z} = \frac{\partial^2 f(x_0, y_0, z_0)}{\partial x\, \partial z}$$

or

$$u''_{x^2} = f''_{x^2}(x_0, y_0, z_0), \quad u''_{xy} = f''_{xy}(x_0, y_0, z_0),$$
$$u''_{xz} = f''_{xz}(x_0, y_0, z_0)^{\dagger}.$$

In an analogous way we define the derivatives of the third, fourth, etc., orders (third, fourth, etc., derivatives). The general definition of the partial derivative of the nth order can be deduced by induction.

Observe that a partial derivative of a higher order taken with respect to different variables, e.g.

$$\frac{\partial^2 u}{\partial x\, \partial y}, \quad \frac{\partial^2 u}{\partial y\, \partial x}, \quad \frac{\partial^4 u}{\partial x\, \partial y\, \partial z^2}, \ldots,$$

is called a mixed derivative.

Examples. (1) Suppose that $u = x^4 y^3 z^2$; then

$$u'_x = 4x^3y^3z^2, \quad u''_{xy} = 12x^3y^2z^2, \quad u'''_{xyz} = 24x^3y^2z, \quad u^{(4)}_{xyzx} = 72x^2y^2z,$$
$$u'_y = 3x^4y^2z^2, \quad u''_{yx} = 12x^3y^2z^2, \quad u'''_{yxx} = 36x^2y^2z^2, \quad u^{(4)}_{yxxz} = 72x^2y^2z,$$
$$u'_z = 2x^4y^3z, \quad u''_{zx} = 8x^3y^3z, \quad u'''_{zxy} = 24x^3y^2z, \quad u^{(4)}_{zxyx} = 72x^2y^2z.$$

(2) We have already considered the partial derivatives of the function $u = \arctan(x/y)$ [Sec. 138, (*2*)]:

$$\frac{\partial u}{\partial x} = \frac{y}{x^2+y^2}, \quad \frac{\partial u}{\partial y} = -\frac{x}{x^2+y^2};$$

we now calculate the higher derivatives:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) = -\frac{2xy}{(x^2+y^2)^2},$$
$$\frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = \frac{x^2-y^2}{(x^2+y^2)^2},$$

---

† Evidently, the differential symbols should be regarded as whole symbols. The square ∂x^2 in the denominator conventionally replaces $\partial x \partial x$ and indicates differentiating twice with respect to x; similarly the index x^2 at the bottom replaces xx. This remark should henceforth be borne in mind.

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial}{\partial x}\left(-\frac{x}{x^2+y^2}\right) = \frac{x^2-y^2}{(x^2+y^2)^2},$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(-\frac{x}{x^2+y^2}\right) = \frac{2xy}{(x^2+y^2)^2};$$

$$\frac{\partial^3 u}{\partial x^2\,\partial y} = \frac{\partial}{\partial y}\left(-\frac{2xy}{(x^2+y^2)^2}\right) = \frac{6xy^2-2x^3}{(x^2+y^2)^3},$$

$$\frac{\partial^3 u}{\partial y\,\partial x^2} = \frac{\partial}{\partial x}\left(\frac{x^2-y^2}{(x^2+y^2)^2}\right) = \frac{6xy^2-2x^3}{(x^2+y^2)^3},$$

etc.

147. Theorems on mixed derivatives. On examining Examples *(1)* and *(2)* of the previous section we observe that mixed derivatives taken with respect to the same variables, but in a different order, are equal.

It should be observed that this by no means necessarily follows from the definition of mixed derivatives; there exist cases when this does not hold.

For instance, consider the function

$$f(x, y) = xy\,\frac{x^2-y^2}{x^2+y^2} \quad (\text{for } x^2+y^2>0), \quad f(0, 0) = 0.$$

We have

$$f'_x(x, y) = y\left[\frac{x^2-y^2}{x^2+y^2} + \frac{4x^2y^2}{(x^2+y^2)^2}\right] \quad (\text{for } x^2+y^2>0),$$

$$f'_x(0, 0) = 0.$$

If we set $x = 0$, for any y (including $y = 0$) we obtain $f'_x(0, y) = -y$. Differentiating this with respect to y we have $f''_{xy}(0, y) = -1$; hence, in particular, at the point (0, 0) we have

$$f''_{xy}(0, 0) = -1.$$

Calculating f''_{yx} in the same way at the point (0, 0) we have

$$f''_{yx}(0, 0) = 1.$$

Thus, for this function $f''_{xy}(0, 0) \neq f''_{yx}(0, 0)$.

Nevertheless the identity of the mixed derivatives, differing only in the order of the differentiation, observed in the above examples is not accidental: it occurs for a wide class of cases.

THEOREM. *Assume that (1) the function $f(x, y)$ is defined in an (open) domain $\mathcal{D}$; (2) there exist in this domain the first derivatives f'_x and f'_y and also the second mixed derivatives f''_{xy} and f''_{yx}, and finally,*

(3) the latter derivatives f''_{xy} and f''_{yx} are continuous functions of x and y at a point (x_0, y_0) of the domain $\mathcal{D}$. Then at this point

$$f''_{xy}(x_0, y_0) = f''_{yx}(x_0, y_0). \tag{1}$$

Proof. Consider the expression

$$W = \frac{f(x_0+h, y_0+k) - f(x_0+h, y_0) - f(x, y_0+k) + f(x_0, y_0)}{hk},$$

where h, k are non-zero (for instance, we assume they are positive) and so small that the whole rectangle $[x_0, x_0+h; y_0, y_0+k]$ is contained in $\mathcal{D}$; only such h and k will be considered below.

Now introduce an auxiliary function of x:

$$\varphi(x) = \frac{f(x,\ y_0+k) - f(x,\ y_0)}{k},$$

which, by *(2)*, has in the interval $[x_0, x_0+h]$ the derivative

$$\varphi'(x) = \frac{f'_x(x,\ y_0+k) - f'_x(x,\ y_0)}{k},$$

and consequently is continuous. With the aid of this function, the expression W, which is equal to

$$W = \frac{1}{h}\left[\frac{f(x_0+h,\ y_0+k) - f(x_0+h,\ y_0)}{k} - \frac{f(x_0,\ y_0+k) - f(x_0,\ y_0)}{k}\right],$$

can be represented in the form

$$W = \frac{\varphi(x_0+h) - \varphi(x_0)}{h}.$$

Since the function $\varphi(x)$ satisfies all the conditions of Lagrange's theorem in the interval $[x_0, x_0+h]$ [Sec. 102], we can transform W, by means of the formula of finite increments, as follows:

$$W = \varphi'(x_0+\theta h) = \frac{f'_x(x_0+\theta h,\ y_0+k) - f'_x(x_0+\theta h,\ y_0)}{k}$$
$$(0<\theta<1).$$

Taking into account the existence of the second derivative $f''_{xy}(x, y)$ we can again apply the formula of finite increments, this time to

the function of $y = f'_x(x_0 + \theta h, y)$ in the interval $[y_0, y_0 + k]$. We finally obtain

$$W = f''_{xy}(x_0 + \theta h,\ y_0 + \theta_1 k) \qquad (0 < \theta,\ \theta_1 < 1). \tag{2}$$

But the expression W contains x and h on the one hand and y and k on the other hand, in the same way. Therefore we may exchange their roles, and introducing the auxiliary function

$$\psi(y) = \frac{f(x_0 + h,\ y) - f(x_0,\ y)}{h},$$

in an analogous way we have

$$W = f''_{yx}(x_0 + \theta_2 h,\ y_0 + \theta_3 k) \qquad (0 < \theta_2,\ \theta_3 < 1). \tag{3}$$

Comparing (2) and (3) we obtain

$$f''_{xy}(x_0 + \theta h,\ y_0 + \theta_1 k) = f''_{yx}(x_0 + \theta_2 h,\ y_0 + \theta_3 k).$$

If now h and k tend to zero we pass to the limit in the last relation. By the boundedness of the factors $\theta, \theta_1, \theta_2, \theta_3$ the arguments on the right and on the left tend to x_0 and y_0, respectively. Then in view of (3) we finally obtain

$$f''_{xy}(x_0,\ y_0) = f''_{yx}(x_0,\ y_0).$$

This completes the proof.

Thus, continuous mixed derivatives f''_{xy} and f''_{yx} are always equal.

In the example examined above these derivatives

$$f''_{xy} = f''_{yx} = \frac{x^2 - y^2}{x^2 + y^2}\left\{1 + \frac{8x^2y^2}{(x^2 + y^2)^2}\right\} \qquad (x^2 + y^2 > 0)$$

have no limit at all when $x \to 0$, $y \to 0$ and consequently have a discontinuity at the point (0, 0). Naturally our theorem cannot be applied to this case.

Remark. A comment on the identity of the mixed derivatives with attempts to prove the result was made first by Euler and Clairaut† in 1740. A strict proof was first given by Schwarz‡ as late as 1873.

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† Alexis Claude Clairaut (1713–1765)—an outstanding French mathematician.

‡ Karl Herman Schwarz (1843–1921)—a German mathematician.

We should note the connection between the problem of changing the order of differentiation and the general problem of changing the order of two limit operations [investigated in Sec. 131].

We have the following general theorem on mixed derivatives:

THEOREM. *Suppose that the function* $u = f(x_1, \ldots, x_m)$ *of* m *arguments is defined in an open* m*-dimensional domain* $\mathcal{D}$*, and has in this domain all possible partial derivatives up to the* $(n-1)$*-th order including the latter, and mixed derivatives of the* n*-th order, all these derivatives being continuous in* $\mathcal{D}$.

Under these conditions the value of any n*-th mixed derivative is independent of the order of differentiation.*

We shall not dwell here on the proof, which is based on the preceding theorem.

Since henceforth we shall always assume the continuity of the derivatives, the order of differentiation will be immaterial. In using a mixed derivative we usually collect the differentiations with respect to the same variable.

148. Differentials of higher orders. Suppose a function $u = f(x_1, \ldots, x_m)$, having continuous partial derivatives of the first order, is given over the domain $\mathcal{D}$. Then the (total) differential du is given by the expression:

$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \ldots + \frac{\partial u}{\partial x_m} dx_m,$$

where $dx_1, \ldots, dx_m$ are arbitrary increments of the independent variables $x_1, \ldots, x_m$.

We observe that du is also a function of $x_1, \ldots, x_m$. If we assume the existence of the continuous partial derivatives of the second order of u, then du has continuous partial derivatives of the first order and we may consider the total differential of the differential du, $d(du)$, which is called the differential of the second order (or the second differential) of u; it is denoted by the symbol d^2u.

It is important to emphasize that the increments $dx_1, \ldots, dx_m$ are now regarded as constant and remain so when passing from one differential to the next (the second differentials $d^2x_1, \ldots, d^2x_m$ are zeros).

Thus, making use of the familiar rules of differentiation [Sec. 143] we have

$$d^2u = d(du) = d\left(\frac{\partial u}{\partial x_1}dx_1 + \frac{\partial u}{\partial x_2}dx_2 + \ldots + \frac{\partial u}{\partial x_m}dx_m\right)$$

$$= d\left(\frac{\partial u}{\partial x_1}\right)dx_1 + d\left(\frac{\partial u}{\partial x_2}\right)dx_2 + \ldots + d\left(\frac{\partial u}{\partial x_m}\right)dx_m$$

or, in full,

$$d^2u = \left(\frac{\partial^2 u}{\partial x_1^2}dx_1 + \frac{\partial^2 u}{\partial x_1 \partial x_2}dx_2 + \ldots + \frac{\partial^2 u}{\partial x_1 \partial x_m}dx_m\right)dx_1$$

$$+ \left(\frac{\partial^2 u}{\partial x_2 \partial x_1}dx_1 + \frac{\partial^2 u}{\partial x_2^2}dx_2 + \ldots + \frac{\partial^2 u}{\partial x_2 \partial x_m}dx_m\right)dx_2 +$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \left(\frac{\partial^2 u}{\partial x_m \partial x_1}dx_1 + \frac{\partial^2 u}{\partial x_m \partial x_2}dx_2 + \ldots + \frac{\partial^2 u}{\partial x_m^2}dx_m\right)dx_m$$

$$= \frac{\partial^2 u}{\partial x_1^2}dx_1^2 + \frac{\partial^2 u}{\partial x_2^2}dx_2^2 + \ldots + \frac{\partial^2 u}{\partial x_m^2}dx_m^2$$

$$+ 2\frac{\partial^2 u}{\partial x_1 \partial x_2}dx_1 dx_2 + 2\frac{\partial^2 u}{\partial x_1 \partial x_3}dx_1 dx_3 + \ldots$$

$$+ 2\frac{\partial^2 u}{\partial x_2 \partial x_3}dx_2 dx_3 + \ldots + 2\frac{\partial^2 u}{\partial x_{m-1} \partial x_m}dx_{m-1}dx_m.$$

In an analogous way we define the differential of the third order d^3u, etc. More generally, the $(n-1)$th differential $d^{n-1}u$ being defined, the differential of the nth order d^nu is defined as the (total) differential of the differential of the $(n-1)$th order

$$d^nu = d(d^{n-1}u).$$

If the function u has continuous partial derivatives of all orders up to and including the nth, then the existence of the nth differential follows. But the full expressions of the latter differentials become more and more complicated. To simplify the symbols we employ the following device.

First, in the expression for the first differential we conventionally "take the letter u outside the brackets"; then it can be written symbolically in the form

$$du = \left(\frac{\partial}{\partial x_1}dx_1 + \frac{\partial}{\partial x_2}dx_2 + \ldots + \frac{\partial}{\partial x_m}dx_m\right)u.$$

We now observe that if in the expression for the second differential we also "take u outside the brackets" then the expression remaining in the brackets is formally the square of the expression

$$\frac{\partial}{\partial x_1}dx_1 + \frac{\partial}{\partial x_2}dx_2 + \ldots + \frac{\partial}{\partial x_m}dx_m;$$

therefore the second differential can be written symbolically as

$$d^2u = \left(\frac{\partial}{\partial x_1}dx_1 + \frac{\partial}{\partial x_2}dx_2 + \ldots + \frac{\partial}{\partial x_m}dx_m\right)^2 u.$$

In an analogous way we can write the third differential, the fourth, etc. This rule is general: for any n we have symbolically

$$d^nu = \left(\frac{\partial}{\partial x_1}dx_1 + \frac{\partial}{\partial x_2}dx_2 + \ldots + \frac{\partial}{\partial x_m}dx_m\right)^n u; \qquad (4)$$

this relation should be remembered as follows: first the "polynomial" in the brackets is formally, in accordance with the rules of algebra, taken to the power n, then all the terms thus obtained are "multiplied" by u (which is written in the numerators following the symbols ∂^n), and then all the symbols are endowed with their meaning as derivatives and differentials.

Rule (4) can be proved by the method of mathematical induction.

Thus, the nth differential is a homogeneous integral polynomial of the nth degree or, we may say, it is a form of the nth degree with respect to the differentials of the independent variables, the coefficients being the partial derivatives of the nth order multiplied by integral constants ("polynomial" coefficients).

For instance, if $u = f(x, y)$, we have

$$d^2u = \frac{\partial^2 u}{\partial x^2}dx^2 + 2\frac{\partial^2 u}{\partial x\,\partial y}dx\,dy + \frac{\partial^2 u}{\partial y^2}dy^2,$$

$$d^3u = \frac{\partial^3 u}{\partial x^3}dx^3 + 3\frac{\partial^3 u}{\partial x^2\,\partial y}dx^2dy + 3\frac{\partial^3 u}{\partial x\,\partial y^2}dx\,dy^2 + \frac{\partial^3 u}{\partial y^3}dy^3,$$

$$d^4u = \frac{\partial^4 u}{\partial x^4}dx^4 + 4\frac{\partial^4 u}{\partial x^3\,\partial y}dx^3\,dy + 6\frac{\partial^4 u}{\partial x^2\,\partial y^2}dx^2\,dy^2 + 4\frac{\partial^4 u}{\partial x\,\partial y^3}dx\,dy^3 + \frac{\partial^4 u}{\partial y^4}dy^4,$$

etc. Setting, for instance, $u = \arctan(x/y)$ we have

$$du = \frac{y\,dx - x\,dy}{x^2+y^2}, \quad d^2u = \frac{2xy\,(dy^2-dx^2) + 2\,(x^2-y^2)\,dx\,dy}{(x^2+y^2)^2},$$

$$d^3u = \frac{(6x^2y-2y^3)dx^3 + (18xy^2-6x^3)dx^2\,dy}{(x^2+y^2)^3} + \frac{(6y^3-18x^2y)dx\,dy^2 + (2x^3-6xy^2)dy^3}{(x^2+y^2)^3},$$

etc.

149. Differentials of compound functions. Consider now the compound function

$$u = f(x_1,\ x_2,\ \ldots,\ x_m),$$

where

$$x_i = \varphi_i(t_1, t_2, \ldots, t_k) \qquad (i = 1, 2, \ldots, m).$$

In this case the first differential may be written in the previous form

$$du = \frac{\partial u}{\partial x_1}dx_1 + \frac{\partial u}{\partial x_2}dx_2 + \ldots + \frac{\partial u}{\partial x_m}dx_m$$

(by the invariance of the form of the first differential, Sec. 143). But here $dx_1, dx_2, \ldots, dx_m$ are differentials not of the independent variables but of functions, and consequently they are functions themselves and may not be constant, unlike the preceding case.

Calculating the second differential of the function we now have (making use of the rules of differentiation given in Sec. 143)

$$\begin{aligned} d^2u &= d\left(\frac{\partial u}{\partial x_1}\right)dx_1 + d\left(\frac{\partial u}{\partial x_2}\right)dx_2 + \ldots + d\left(\frac{\partial u}{\partial x_m}\right)dx_m \\ &\quad + \frac{\partial u}{\partial x_1}d(dx_1) + \frac{\partial u}{\partial x_2}d(dx_2) + \ldots + \frac{\partial u}{\partial x_m}d(dx_m) \\ &= \left(\frac{\partial}{\partial x_1}dx_1 + \frac{\partial}{\partial x_2}dx_2 + \ldots + \frac{\partial}{\partial x_m}dx_m\right)^2 u \\ &\quad + \frac{\partial u}{\partial x_1}d^2x_1 + \frac{\partial u}{\partial x_2}d^2x_2 + \ldots + \frac{\partial u}{\partial x_m}d^2x_m. \end{aligned}$$

We observe that for a differential of order higher than the first the form is not in general invariant.

Consider now the particular case when $x_1, \ldots, x_m$ are linear functions of $t_1, \ldots, t_k$, i.e. when

$$x_i = \alpha_i^{(1)} t_1 + \alpha_i^{(2)} t_2 + \ldots + \alpha_i^{(m)} t_m + \beta_i \qquad (i = 1, 2, \ldots, m),$$

where $\alpha_i^{(j)}$ and β_i are constants.

In this case we have

$$dx_i = \alpha_i^{(1)} dt_1 + \ldots + \alpha_i^{(m)} dt_m = \alpha_i^{(1)} \Delta t_1 + \ldots + \alpha_i^{(m)} \Delta t_m.$$

We observe that the first differentials of the functions $x_1, \ldots, x_m$ are now constant, i.e. they are independent of $t_1, \ldots, t_k$; consequently the whole argument of Sec. 148 is applicable. This fact implies that when replacing the independent variables $x_1, \ldots, x_m$ by linear functions of new variables $t_1, \ldots, t_k$, the previous expressions may be preserved even for differentials of higher orders. In these expressions the differentials $dx_1, \ldots, dx_m$ are identical with the increments $\Delta x_1, \ldots, \Delta x_m$ but these increments are not arbitrary and vary in a manner depending on the increments $\Delta t_1, \ldots, \Delta t_k$.

This simple but important remark is due to Cauchy; it will be employed in the following section.

150. The Taylor formula. We know [Sec. 107, (12b)] that a function $F(t)$, provided its first $n+1$ derivatives exist, can be expanded into the Taylor series in the following way:

$$\Delta F(t_0) = dF(t_0) + \frac{1}{2!} d^2F(t_0) + \ldots + \frac{1}{n!} d^nF(t_0) + \frac{1}{(n+1)!} d^{n+1}F(t_0 + \theta \Delta t) \qquad (0 < \theta < 1).$$

It is important to observe that the quantity dt, which appears to various powers in the expressions of the differentials on the right, is equal to the increment Δt which appears in the increment of the function on the left:

$$\Delta F(t_0) = F(t_0 + \Delta t) - F(t_0).$$

In exactly the last form the Taylor formula is extended to the case of functions of several variables (Cauchy).

To simplify the notation we confine ourselves to a function of two variables $f(x, y)$.

Assume that in the neighbourhood of a point (x_0, y_0), this function has continuous derivatives of all orders up to and including the $(n+1)$th. Let x, y have increments $\Delta x, \Delta y$ at $x = x_0, y = y_0$ such that the segment of straight line connecting the points (x_0, y_0) and $(x_0 + \Delta x, y_0 + \Delta y)$ does not leave the considered neighbourhood of the point (x_0, y_0).

It is required to prove that, under the above assumptions concerning the function $f(x, y)$, the following relation holds:

$$\begin{aligned}\Delta f(x_0, y_0) &= f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) \\ &= df(x_0, y_0) + \frac{1}{2!} d^2 f(x_0, y_0) + \ldots + \frac{1}{n!} d^n f(x_0, y_0) \\ &\quad + \frac{1}{(n+1)!} d^{n+1} f(x_0 + \theta \Delta x, y_0 + \theta \Delta y) \\ &\qquad (0 < \theta < 1); \qquad (5)\end{aligned}$$

the differentials dx and dy in the various powers entering the expression on the right are equal to the increments of the independent variables Δx and Δy which resulted in the increment of the function on the left.

To prove this assertion we introduce a new independent variable t setting

$$x = x_0 + t\Delta x, \quad y = y_0 + t\Delta y \quad (0 \leqslant t \leqslant 1). \qquad (6)$$

Substituting these values of x and y in the function $f(x, y)$ we arrive at the compound function of one variable t:

$$F(t) = f(x_0 + t\Delta x, y_0 + t\Delta y).$$

We know that the formulae (6) represent, geometrically, the segment of the straight line joining the points $M_0(x_0, y_0)$ and $M_1(x_0 + \Delta x, y_0 + \Delta y)$.

It is evident that, instead of the increment

$$\Delta f(x_0, y_0) = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0),$$

we may consider the increment of the auxiliary function

$$\Delta F(0) = F(1) - F(0),$$

since the two increments are equal. But $F(t)$ is a function of one variable and has $(n+1)$ continuous derivatives; consequently we may apply to it the deduced Taylor formula; thus we obtain

$$\Delta F(0) = F(1) - F(0) = dF(0) + \frac{1}{2!} d^2F(0) + \ldots$$

$$+ \frac{1}{n!} d^nF(0) + \frac{1}{(n+1)!} d^{n+1}F(\theta) \quad (0 < \theta < 1), \quad (7)$$

the differential dt entering in various powers the expression on the right being equal to $\Delta t = 1 - 0 = 1$.

Now, making use of the fact that in a linear change of variables the property of invariance of the form holds for higher differentials, we have

$$dF(0) = f'_x(x_0, y_0)dx + f'_y(x_0, y_0)dy = df(x_0, y_0),$$

$$d^2F(0) = f''_{x^2}(x_0, y_0)dx^2 + 2f''_{xy}(x_0, y_0)dx\,dy + f''_{y^2}(x_0, y_0)dy^2 = d^2f(x_0, y_0),$$

etc. Finally for the $(n+1)$th differential we have

$$d^{n+1}F(\theta) = d^{n+1}f(x_0 + \theta\Delta x, y_0 + \theta\Delta y).$$

It is important to note that here the differentials dx and dy do not differ from the previously considered increments Δx and Δy. In fact, since $dt = 1$,

$$dx = \Delta x\,dt = \Delta x, \quad dy = \Delta y\,dt = \Delta y.$$

Substituting this into the expansion (7) we arrive at the required expansion (5).

The reader should realize that although in the differential form the Taylor formula for the functions of several variables has as simple a form as in the case of one variable, the full expression is much more complicated.

§ 3. Extrema, the greatest and the smallest values

151. Extrema of functions of several variables. Necessary conditions. Suppose that the function

$$u = f(x_1, x_2, \ldots, x_m)$$

is defined in a domain $\mathcal{D}$ and $(x_1^0, \ldots, x_m^0)$ is an interior point of the domain.

We say that the function $f(x_1, \ldots, x_m)$ has a maximum (minimum) at the point $(x_1^0, \ldots, x_m^0)$ if it can be surrounded by a neighbourhood

$$(x_1^0 - \delta_1,\ x_1^0 + \delta_1;\ x_2^0 - \delta_2,\ x_2^0 + \delta_2;\ \ldots;\ x_m^0 - \delta_m,\ x_m^0 + \delta_m),$$

such that for all points of this neighbourhood the inequality

$$f(x_1,\ x_2, \ldots,\ x_m) \leqslant f(x_1^0,\ x_2^0, \ldots,\ x_m^0)$$
$$(\geqslant)$$

holds.

If this neighbourhood is taken sufficiently small, so that the equality sign may be excluded, i.e. so that at all points except $(x_1^0, \ldots, x_m^0)$ itself the strict inequality

$$f(x_1,\ x_2, \ldots, x_m) < f(x_1^0,\ x_2^0, \ldots,\ x_m^0)$$
$$(>)$$

is satisfied, then we say that a *proper* maximum (minimum) occurs at the point $(x_1^0, \ldots, x_m^0)$; otherwise the maximum (minimum) is said to be *improper.*

To denote a maximum or minimum we use the common term—an *extremum.*

We shall prove that if the finite partial derivatives

$$f'_{x_1}(x_1^0, \ldots, x_m^0), \ldots, f'_{x_m}(x_1^0, \ldots, x_m^0)$$

exist at this point, then all these partial derivatives vanish, and thus the vanishing of the partial derivatives of the first order is a necessary condition for the existence of an extremum.

For this purpose set $x_2 = x_2^0, \ldots, x_m = x_m^0$ regarding x_1 as variable; thus we have a function of one variable x_1:

$$u = f(x_1, x_2^0, \ldots, x_m^0).$$

Since we have assumed that an extremum exists at the point $(x_1^0, \ldots, x_m^0)$ (for definiteness suppose it is a maximum), it follows in particular that in a neighbourhood $(x_1^0 - \delta_1, x_1^0 + \delta_1)$ of the point $x_1 = x_1^0$ the inequality

$$f(x_1, x_2, \ldots, x_m^0) \leqslant f(x_1^0, x_2^0, \ldots, x_m^0)$$

must be satisfied. Hence the above function of one variable has a maximum at the point $x_1 = x_1^0$ and consequently, by Fermat's theorem [Sec. 100], we have

$$f'_{x_1}(x_1^0,\ x_2^0, \ldots, x_m^0) = 0.$$

In the same way we can prove that the other partial derivatives also vanish at the point $(x_1^0, \ldots, x_m^0)$.

Thus, the "suspect" points are those at which the partial derivatives of the first order vanish; their coordinates may be found by solving the system of equations

$$\left.\begin{aligned} f'_{x_1}(x_1, x_2, \ldots, x_m) &= 0, \\ f'_{x_2}(x_1, x_2, \ldots, x_m) &= 0, \\ \cdots\cdots\cdots\cdots & \\ f'_{x_m}(x_1, x_2, \ldots, x_m) &= 0. \end{aligned}\right\} \tag{1}$$

As in the case of one variable such points are called *stationary*.

152. Investigation of stationary points (for the case of two variables). As in the case of a function of one variable, an extremum does not occur at every stationary point. Considering, for instance,

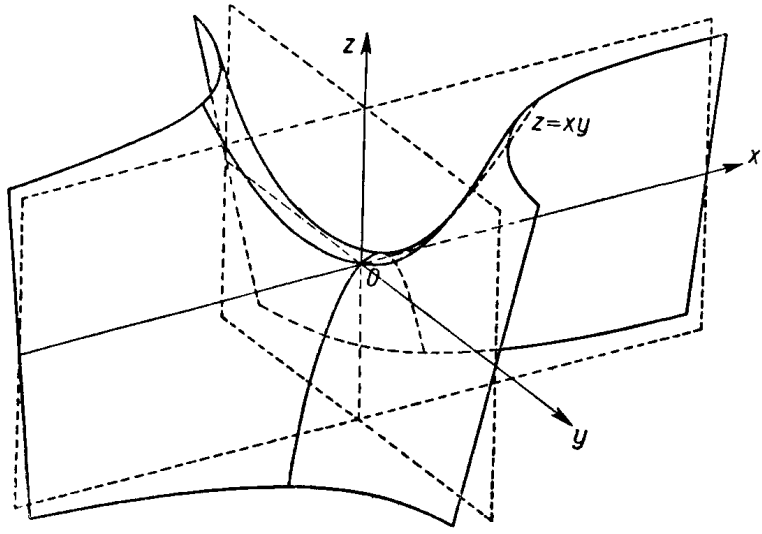

FIG. 59.

the simple function $z = xy$, we have $z'_x = y$ and $z'_y = x$ which vanish simultaneously at only one point—the origin $(0, 0)$—at which $z = 0$. However, it is clear that in any vicinity of this point the function takes both positive and negative values and there is thus no extremum. Figure 59 represents the surface (a hyperbolic para-

boloid) expressed by the equation $z = xy$; near the origin it has the form of a saddle, bending upwards in one vertical plane and downwards in the perpendicular vertical plane.

Thus the question arises as to sufficient conditions for the existence (or absence) of an extremum, i.e. further investigation of a stationary point.

We confine ourselves to a function of two variables, $f(x, y)$. We assume that the function is defined, continuous and has continuous partial derivatives of the first and second orders in the neighbourhood of (x_0, y_0), which is a stationary point, i.e. it satisfies the conditions

$$f'_x(x_0, y_0) = 0, \quad f'_y(x_0, y_0) = 0. \tag{1a}$$

In order to establish whether or not the function has an extremum at the point (x_0, y_0) it is natural to examine the difference

$$\Delta = f(x, y) - f(x_0, y_0).$$

We expand this by the Taylor series with the remainder term in Lagrange's form [Sec. 150, (5)], confining ourselves to two terms. Since (x_0, y_0) is assumed to be a stationary point, the first term vanishes and we have

$$\Delta = \frac{1}{2!}\{f''_{x^2}\Delta x^2 + 2f''_{xy}\Delta x\Delta y + f''_{y^2}\Delta y^2\}. \tag{2}$$

Now the role of the increments $\Delta x, \Delta y$ is played by the differences $x - x_0, y - y_0$ and the derivatives are calculated at a point

$$(x_0 + \theta\Delta x,\ y_0 + \theta\Delta y).$$

Now introduce the values of these derivatives at the point (x_0, y_0),

$$a_{11} = f''_{x^2}(x_0, y_0), \quad a_{12} = f''_{xy}(x_0, y_0), \quad a_{22} = f''_{y^2}(x_0, y_0), \tag{3}$$

and set

$$f''_{x^2}(x_0 + \theta\Delta x, y_0 + \theta\Delta y) = a_{11} + \alpha_{11},$$
$$f''_{xy}(\ldots) = a_{12} + \alpha_{12}, \quad f''_{y^2}(\ldots) = a_{22} + \alpha_{22}.$$

Hence by the continuity of the second derivatives

$$\text{all} \quad \alpha \to 0 \quad \text{as} \quad \Delta x \to 0,\ \Delta y \to 0. \tag{4}$$

The difference Δ can be written in the form

$$\Delta = \frac{1}{2}\{a_{11}\Delta x^2 + 2a_{12}\Delta x\Delta y + a_{22}\Delta y^2 + \alpha_{11}\Delta x^2 + 2\alpha_{12}\Delta x\Delta y + \alpha_{22}\Delta y^2\}.$$

We shall establish that the behaviour of the difference Δ essentially depends on the sign of the expression $a_{11}a_{22} - a_{12}^2$.

To simplify the reasoning we now assume that $\Delta x = \varrho\cos\varphi$, $\Delta y = \varrho\sin\varphi$ where $\varrho = \sqrt{(\Delta x^2 + \Delta y^2)}$ is the distance between the points (x_0, y_0) and (x, y). Now finally

$$\Delta = \frac{\varrho^2}{2}\{a_{11}\cos^2\varphi + 2a_{12}\cos\varphi\sin\varphi + a_{22}\sin^2\varphi + \alpha_{11}\cos^2\varphi + 2\alpha_{12}\cos\varphi\sin\varphi + \alpha_{22}\sin^2\varphi\}.$$

(1) Suppose, first, that $a_{11}a_{22} - a_{12}^2 > 0$.

In this case $a_{11}a_{22} > 0$, whence $a_{11} \neq 0$, and the first three terms in the curly brackets can be written in the form

$$\frac{1}{a_{11}}[(a_{11}\cos\varphi + a_{12}\sin\varphi)^2 + (a_{11}a_{22} - a_{12}^2)\sin^2\varphi]. \tag{5}$$

It is now clear that the expression in the square brackets is always positive and therefore the polynomial consisting of the above three terms does not vanish for any value of φ and has the same sign as the coefficient a_{11}. Since its absolute value as a function of φ is continuous in the interval $[0, 2\pi]$, it is bounded below by some m:

$$|a_{11}\cos^2\varphi + 2a_{12}\cos\varphi\sin\varphi + a_{22}\sin^2\varphi| \geqslant m > 0.$$

On the other hand, considering the last three terms in the curly brackets we find, by (4),

$$|\alpha_{11}\cos^2\varphi + 2\alpha_{12}\cos\varphi\sin\varphi + \alpha_{22}\sin^2\varphi| \leqslant |\alpha_{11}| + 2|\alpha_{12}| + |\alpha_{22}| < m$$

for all φ, provided only that ϱ (and hence also Δx, Δy) is sufficiently small. But then the whole expression in the curly brackets, and hence the difference Δ as well, has the same sign as the first polynomial, i.e. the sign of a_{11}.

Thus if $a_{11} > 0$, then also $\Delta > 0$, i.e. the function has a minimum at the point (x_0, y_0), while if $a_{11} < 0$ we have also $\Delta < 0$ and hence there is a maximum.

(2) Now suppose that $a_{11}a_{22} - a_{12}^2 < 0$.

Consider the case when $a_{11} \neq 0$; then we can again use the transformation (5). For $\varphi = \varphi_1 = 0$ the expression in the square brackets is positive, for it is equal to a_{11}^2. Conversely, if we determine $\varphi = \varphi_2$ from the condition

$$a_{11}\cos\varphi_2 + a_{12}\sin\varphi_2 = 0 \qquad (\sin\varphi_2 \neq 0),$$

the expression reduces to $(a_{11}a_{22} - a_{12}^2)\sin^2\varphi_2$ and is negative. For ϱ sufficiently small the second polynomial in the curly brackets, both for $\varphi = \varphi_1$ and for $\varphi = \varphi_2$, is arbitrarily small and the sign of Δ is determined by the sign of the first polynomial. Thus, in an arbitrarily small neighbourhood of the point (x_0, y_0) on the rays determined by the angles $\varphi = \varphi_1$ and $\varphi = \varphi_2$, the difference Δ has values of opposite signs. Consequently, there is no extremum at this point.

If $a_{11} = 0$ and the first polynomial in the curly brackets is reduced to

$$2a_{12}\cos\varphi\sin\varphi + a_{22}\sin^2\varphi = \sin\varphi(2a_{12}\cos\varphi + a_{22}\sin\varphi),$$

then, making use of the fact that $a_{12} \neq 0$, we can find an angle $\varphi_1 \neq 0$ such that

$$|a_{22}|\,|\sin\varphi_1| < 2|a_{12}|\,|\cos\varphi_1|;$$

then for $\varphi = \varphi_1$ and $\varphi = \varphi_2 = -\varphi_1$ the considered polynomial consisting of three terms has opposite signs and this proves the assertion.

Thus, if $a_{11}a_{22} - a_{12}^2 > 0$ at the stationary point (x_0, y_0) the function $f(x, y)$ has an extremum, namely a maximum for $a_{11} < 0$ and a minimum for $a_{11} > 0$. If $a_{11}a_{22} - a_{12}^2 < 0$ there is no extremum.

In the case $a_{11}a_{22} - a_{12}^2 = 0$, to solve the problem we have to consider the higher derivatives; this "doubtful" case will not be dealt with here.

Remark. Euler was the first to note the necessity of the conditions

$$f_x'(x_0, y_0) = 0, \quad f_y'(x_0, y_0) = 0$$

in order that the function $f(x, y)$ should have a maximum at the point (x_0, y_0). However, he wrongly assumed that the presence for the function of an extremum of the same kind with respect to each variable separately (which will occur, for instance, when the derivatives f_{x^2}'', f_{y^2}'' have the same sign) is a sufficient

condition for it to have an extremum at that point. Lagrange noticed Euler's mistake and he established the inequality

$$f''_{x^2} f''_{y^2} - (f''_{xy})^2 > 0$$

as a sufficient condition. He also indicated that the converse inequality establishes the absence of an extremum, but he did not justify this completely.

Examples. (1) Let us investigate the maximum and minimum of the function

$$z = \frac{x^2}{2p} + \frac{y^2}{2q} \quad (p>0,\ q>0).$$

Calculate the partial derivatives

$$z'_x = \frac{x}{p}, \quad z'_y = \frac{y}{q}.$$

We see at once that the only stationary point is the origin (0, 0).

Calculating a_{11}, a_{12} and a_{22} we obtain

$$a_{11} = \frac{1}{p}, \quad a_{12} = 0, \quad a_{22} = \frac{1}{q}.$$

Hence $a_{11}a_{22} - a_{12}^2 > 0$. Consequently at the point (0, 0) the function z has a minimum, which incidentally would be clear from a direct investigation.

The geometric interpretation of the function is an elliptic paraboloid with vertex at the origin (see Fig. 55 on p. 233).

(2)
$$z = \frac{x^2}{2p} - \frac{y^2}{2q} \quad (p>0,\ q>0).$$

We have here

$$z'_x = \frac{x}{p}, \quad z'_y = -\frac{y}{q}.$$

Again the stationary point is the origin (0, 0)

We have

$$a_{11} = \frac{1}{p}, \quad a_{12} = 0, \quad a_{22} = -\frac{1}{q},$$

whence $a_{11}a_{22} - a_{12}^2 < 0$. Consequently there is no extremum.

The geometric interpretation here is a hyperbolic paraboloid with vertex at the origin.

(3)
$$z = y^2 + x^4 \quad \text{or} \quad z = y^2 + x^3;$$

in both cases the stationary point is (0, 0) and $a_{11}a_{22} - a_{12}^2 = 0$.

Our criterion does not solve this problem; however, it can be seen directly that in the first case we have a minimum, while in the second there is no extremum.

153. The smallest and the greatest values of a function. Examples. Suppose that a function $u = f(x_1, \ldots, x_m)$ is defined, over a bounded

closed domain $\mathcal{D}$ over which it is continuous and has finite partial derivatives. According to the Weierstrass theorem [Sec. 136], a point $(x_1^0, \ldots, x_m^0)$ can be found in this domain at which the function attains a greatest (smallest) value. If the point $(x_1^0, \ldots, x_m^0)$ is located inside the domain $\mathcal{D}$ it is evident that there the function has a maximum (minimum), and therefore this point is certainly among the "suspect" stationary points. However, the function can attain its greatest (smallest) value on the boundary of the domain as well. Consequently, in order to find the greatest (smallest) value of the function $u = f(x_1, \ldots, x_m)$ in a domain $\mathcal{D}$, it is necessary to find all the "suspect" interior stationary points to compute the values of the function at these points and then to compare them with the values of the function at the boundary points of the domain; the greatest (smallest) of all these values is the greatest (smallest) value of the function in the whole domain.

We elucidate the above discussion by some examples.

(1) We seek the greatest value of the function

$$u = \sin x + \sin y - \sin(x + y)$$

in the triangle bounded by the x-axis, y-axis and the straight line $x + y = 2\pi$ (Fig. 60). We have

$$u'_x = \cos x - \cos(x + y), \quad u'_y = \cos y - \cos(x + y).$$

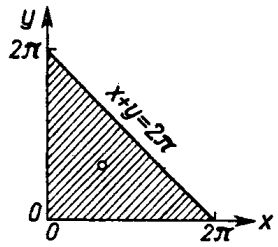

FIG. 60.

Inside the domain the derivatives vanish only at the point $(2\pi/3, 2\pi/3)$ where $u = 3\sqrt{3}/2$. Since the function vanishes on the boundary of the domain, i.e. on the straight lines $x = 0$, $y = 0$ and $x + y = 2\pi$, it is evident that the function has its greatest value at the point $(2\pi/3, 2\pi/3)$.

(2) We seek the greatest value of the product

$$u = xyzt$$

of the non-negative numbers x, y, z, t under the condition that their sum has the constant value

$$x+y+z+t=4c.$$

It will be proved that the greatest value of u is obtained when all the factors are equal, i.e.

$$x=y=z=t=c\dagger.$$

Determining t from the given condition, $t=4c-x-y-z$, we substitute it into u and then

$$u=xyz(4c-x-y-z).$$

Thus we have here a function of three *independent* variables x, y, z in a three-dimensional domain determined by the conditions

$$x\geqslant 0,\quad y\geqslant 0,\quad z\geqslant 0,\quad x+y+z\leqslant 4c.$$

The geometric interpretation of the domain is a tetrahedron bounded by the planes $x=0$, $y=0$, $z=0$, $x+y+z=4c$.

We calculate the derivatives and equate them to zero

$$\frac{\partial u}{\partial x}=yz(4c-2x-y-z)=0,\quad \frac{\partial u}{\partial y}=zx(4c-x-2y-z)=0,$$

$$\frac{\partial u}{\partial z}=xy(4c-x-y-2z)=0.$$

Inside the domain these equations are satisfied only at the point $x=y=z=c$ where $u=c^4$. Since $u=0$ on the boundary of the domain, the function does, in fact, attain its greatest value at the determined point.

Our assertion is proved, for when $x=y=z=c$ we also have $t=c\ddagger$.

Remark. In the given example there is only one stationary point inside the considered domain. We can prove that at this point a maximum occurs. However, in contrast with the result for the function of one variable [see Sec. 118, Remark] we cannot infer from this fact alone that we have found the greatest value of the function in the domain.

The following simple example indicates that such an assertion can, in fact, lead to incorrect results. Consider the function

$$u=x^3-4x^2+2xy-y^2$$

---

† For the sake of definiteness only we have taken the number of factors equal to four. The result is the same for an arbitrary number of factors.

‡ Our reasoning implies that the product $xyzt$ of four positive numbers the sum of which is $4c$ does not exceed c^4 and hence

$$\sqrt[4]{(xyzt)}\leqslant c=\frac{x+y+z+t}{4},$$

i.e. the geometric mean does not exceed the arithmetic mean. This is true for an arbitrary number of numbers.

defined over the rectangle $[-5, 5; -1, 1]$. Its derivatives

$$u'_x = 3x^2 - 8x + 2y, \quad u'_y = 2x - 2y$$

vanish only at the point $(0, 0)$ of this domain. It can easily be proved by means of the criterion of Sec. 152 that the function has a maximum (equal to zero) at this point. However, this is not the greatest value in the domain, since, for instance, at point $(5, 0)$ the value of the function is 25.

Thus we see that, in the case of a function of several variables (when seeking the greatest and the smallest values of the function over a domain), the investigation of maximum and minimum is practically useless.

154. Problems. Many problems both from the field of mathematics and from other fields of science and engineering lead to the problem of determining the greatest or smallest values of a function.

The solutions of problems *(1)* and *(2)* are connected with the procedures examined in the preceding section.

(1) It is required to find among all the triangles which can be inscribed within a circle of radius R the one whose area is the greatest (Fig. 61).

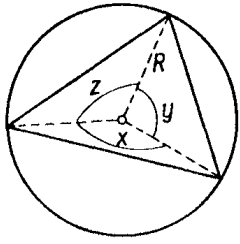

FIG. 61.

Denoting by x, y, z the angles subtended at the centre by the sides of the triangle we have $x + y + z = 2\pi$. Hence $z = 2\pi - x - y$. The area P of the triangle is given by the formula

$$P = \tfrac{1}{2}R^2 \sin x + \tfrac{1}{2}R^2 \sin y + \tfrac{1}{2}R^2 \sin z = \tfrac{1}{2}R^2 [\sin x + \sin y - \sin (x + y)].$$

The domain of variation of the variables x and y is defined by the conditions $x \geqslant 0$, $y \geqslant 0$, $x + y \leqslant 2\pi$. It is required to find the values of the variables for which the expression in square brackets has the greatest value.

We already know [Sec. 153, *(1)*] that these are $x = y = 2\pi/3$ and hence $z = 2\pi/3$; thus, we have obtained an equilateral triangle.

(2) It is required to find that triangle of the set of all triangles of given perimeter $2p$ whose area P is the greatest.

Denote the sides of the triangle by x, y, z; then we have

$$P = \sqrt{[p(p-x)(p-y)(p-z)]}.$$

Setting $z = 2p - x - y$ we could transform P to the form

$$P = \sqrt{[p(p-x)(p-y)(2p-x-y)]}$$

and seek the greatest value of this function in the triangular domain considered already in Sec. 124, *(5)*.

We shall proceed in a different way. The problem is reduced to the determination of the greatest value of the product of positive numbers

$$u = (p-x)(p-y)(p-z)$$

under the condition that their sum is constant, i.e.

$$(p-x)+(p-y)+(p-z) = 3p-2p = p.$$

But we know already [Sec. 153, *(2)*] that all factors in this case are equal, i.e. $x = y = z = 2p/3$. Thus, we again obtain an equilateral triangle.

(3) Consider an electric supply network with the connections in parallel. Figure 62 represents the system, A and B being the contacts of the source of current and $P_1, \ldots, P_n$ the receivers of the current, the corresponding currents being $i_1, \ldots, i_n$. It is required, for a prescribed total potential difference $2e$ in the system, to determine the cross-sections of the conductors so that the smallest possible amount of copper is required for the whole system.

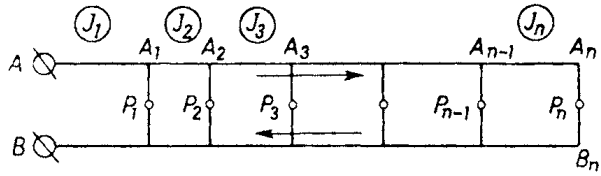

FIG. 62.

Obviously, it is sufficient to examine one of the conductors, say AA_n, since the considerations for the others are the same. Denote by $l_1, \ldots, l_n$ the lengths of the parts $AA_1, \ldots, AA_n$ (in metres) and by $q_1, \ldots, q_n$ the areas of their cross-sections (in square millimetres). Then the expression

$$u = l_1q_1 + l_2q_2 + \ldots + l_nq_n$$

represents the volume of the copper used in the system (in cubic centimetres); we have to find its smallest value, taking into account that the total difference of potential in the conductor AA_n is equal to e.

It is easy to find the currents $J_1, \ldots, J_n$ in the segments $AA_1, \ldots, AA_n$ of the system, namely

$$J_1 = i_1 + i_2 + \ldots + i_n, \quad J_2 = i_2 + \ldots + i_n, \quad J_n = i_n.$$

Denoting by ϱ the resistance of the copper conductor of length 1 metre and cross-section 1 mm², the resistances of the segments are the following:

$$r_1 = \frac{\varrho l_1}{q_1}, \quad r_2 = \frac{\varrho l_2}{q_2}, \quad \ldots, \quad r_n = \frac{\varrho l_n}{q_n}.$$

Hence, by Ohm's law the corresponding potential differences in these segments are

$$e_1 = r_1 J_1 = \frac{\varrho l_1 J_1}{q_1},\ e_2 = r_2 J_2 = \frac{\varrho l_2 J_2}{q_2},\ \ldots,\ e_n = r_n J_n = \frac{\varrho l_n J_n}{q_n}.$$

To avoid complicated calculations, instead of the variables $q_1, \ldots, q_n$ we introduce the quantities $e_1, \ldots, e_n$ connected by the simple condition $e_1 + e_2 + \ldots + e_n = e$, whence $e_n = e - e_1 - e_2 - \ldots - e_{n-1}$.
Then we have

$$q_1 = \frac{\varrho l_1 J_1}{e_1},\quad q_2 = \frac{\varrho l_2 J_2}{e_2},\ \ldots,\ q_n = \frac{\varrho l_n J_n}{e_n} = \frac{\varrho l_n J_n}{e - e_1 - e_2 - \ldots - e_{n-1}}$$

and

$$u = \varrho \left[\frac{l_1^2 J_1}{e_1} + \frac{l_2^2 J_2}{e_2} + \ldots + \frac{l_{n-1}^2 J_{n-1}}{e_{n-1}} + \frac{l_n^2 J_n}{e - e_1 - e_2 - \ldots - e_{n-1}} \right],$$

the domain of variation of the independent variables $e_1, \ldots, e_{n-1}$ being defined by the inequalities

$$e_1 > 0,\ e_2 > 0,\ \ldots,\ e_{n-1} > 0,\ e_1 + e_2 + \ldots + e_{n-1} < e.$$

Equating to zero the derivatives of u with respect to all the variables we obtain the system of equations

$$\begin{aligned}
-\frac{l_1^2 J_1}{e_1^2} + \frac{l_n^2 J_n}{(e - e_1 - \ldots - e_{n-1})^2} &= 0, \\
-\frac{l_2^2 J_2}{e_2^2} + \frac{l_n^2 J_n}{(e - e_1 - \ldots - e_{n-1})^2} &= 0, \ \ldots, \\
-\frac{l_{n-1}^2 J_{n-1}}{e_{n-1}^2} + \frac{l_n^2 J_n}{(e - e_1 - \ldots - e_{n-1})^2} &= 0,
\end{aligned}$$

whence (again introducing e_n)

$$\frac{l_1^2 J_1}{e_1^2} = \frac{l_2^2 J_2}{e_2^2} = \ldots = \frac{l_n^2 J_n}{e_n^2}.$$

It is convenient to denote the common value of the above ratios by $1/\lambda^2$ $(\lambda > 0)$. Then

$$e_1 = \lambda l_1 \sqrt{J_1},\ e_2 = \lambda l_2 \sqrt{J_2},\ \ldots,\ e_n = \lambda l_n \sqrt{J_n},$$

λ being easily determined from the condition $e_1 + \ldots + e_n = e$,

$$\lambda = \frac{e}{l_1 \sqrt{J_1} + l_2 \sqrt{J_2} + \ldots + l_n \sqrt{J_n}}.$$

Finally, returning to the variables $q_1, \ldots, q_n$ we find that

$$q_1 = \frac{\varrho}{\lambda} \sqrt{J_1},\quad q_2 = \frac{\varrho}{\lambda} \sqrt{J_2},\ \ldots,\ q_n = \frac{\varrho}{\lambda} \sqrt{J_n},$$

Therefore, the most economic cross-section of the conductor is proportional to the square root of its current.

Remark. Since the domain of variation of the variables $e_1, \ldots, e_{n-1}$ is open, the second Weierstrass theorem [Sec. 136] cannot be applied directly. However, the boundary of the domain is given by the relations

$$e_1 \geqslant 0, \; e_2 \geqslant 0, \; \ldots, \; e_{n-1} \geqslant 0, \quad e_1 + e_2 + \ldots + e_{n-1} \leqslant e,$$

and in at least one case the equality sign occurs. Thus, when the point $(e_1, \ldots, e_{n-1})$ approaches the boundary, the quantity u tends to infinity. This implies that the determined values $e_1, \ldots, e_{n-1}$ in fact provide the function u with its smallest value.

CHAPTER 10

PRIMITIVE FUNCTION (INDEFINITE INTEGRAL)

§ 1. Indefinite integral and simple methods for its evaluation

155. The concept of a primitive function (and of an indefinite integral). In many problems of science and engineering we encounter the problem of finding a function knowing its derivative.

In Sec. 78, assuming the equation of motion $s = f(t)$ to be known (i.e. the law of change of the distance with time), by differentiation we found the velocity $v = ds/dt$ and then the acceleration $a = dv/dt$. However, it is frequently necessary to solve the inverse problem: the acceleration a is known as a function of time t, $a = a(t)$, and it is required to determine the velocity v and the distance s traversed as functions of the time t. Thus we have to find the function $v = v(t)$ knowing the function $a = a(t)$, a being its derivative. Next, knowing the function v it is required to determine the function $s = s(t)$ for which v is the derivative.

Similarly, knowing the mass $m = m(x)$ continuously distributed over a segment of straight line $[0, x]$ of the x-axis, we found by differentiation [Sec. 78] the "linear" density $\varrho = \varrho(x)$. Naturally, the question arises of whether it is possible to find the magnitude of the distributed mass knowing the law of variation of the density $\varrho = \varrho(x)$, i.e. from a known function $\varrho(x)$ we have to find the function $m = m(x)$ of which ϱ is the derivative.

The function $F(x)$, over the interval $\mathcal{X}$, is called the *primitive* or *primitive function*† for $f(x)$, or the *integral* of $f(x)$, if over the whole interval $f(x)$ is the derivative of the function $F(x)$ or, equivalently,

† The term "primitive function" was introduced by Lagrange (see the footnote on p. 145).

$f(x)dx$ is the differential of $F(x)$

$$F'(x) = f(x) \quad \text{or} \quad dF(x) = f(x)dx^{\dagger}.$$

The determination of all the primitive functions for a function, is called integration and is one of the basic problems of integral calculus; we see that this problem is the inverse to the problem of the differential calculus.

THEOREM. *If in an interval $\mathfrak{X}$ (finite or infinite, closed or otherwise) the function $F(x)$ is a primitive for a function $f(x)$, then so is the function $F(x) + C$ where C is an arbitrary constant. Conversely, every primitive function for $f(x)$ in the interval $\mathfrak{X}$ can be represented in this form.*

Proof. It is evident that if $F(x)$ is a primitive, so also is $F(x) + C$, since $[F(x) + C]' = F'(x) = f(x)$.

Suppose now that $\Phi(x)$ is an arbitrary primitive for $f(x)$ so that we have

$$\Phi'(x) = f(x)$$

over the interval $\mathfrak{X}$. Since the functions $F(x)$ and $\Phi(x)$ have the same derivative over $\mathfrak{X}$, they differ by a constant [Sec. 110, Corollary]:

$$\Phi(x) = F(x) + C.$$

This completes the proof.

It follows from this theorem that it is sufficient to find just one primitive function $F(x)$ for a given function $f(x)$ in order to know all the primitive functions, since they differ by a constant.

Consequently the expression $F(x) + C$, where C is an arbitrary constant, is the general form of the function which has the derivative $f(x)$ or the differential $f(x)dx$. This expression is called the *indefinite integral* of $f(x)$ and is denoted by

$$\int f(x)\,dx,$$

which implicitly contains the arbitrary constant. The function $f(x)$ is called the *integrand* and the product $f(x)dx$ the *integral expression.*

Example. Suppose that $f(x) = x^2$; then it is readily observed that the indefinite integral of this function is

$$\int x^2 dx = \frac{x^3}{3} + C.$$

This can easily be verified by the inverse operation of differentiation.

† In this case it is also said that function $F(x)$ is the primitive (or the integral) for the differential expression $f(x)\,dx$.

We draw the reader's attention to the fact that under the "integral" sign $\int$ we write the differential of the unknown primitive function, not the derivative (in our example $x^2\,dx$, not x^2). This form of notation is historical; it will be explained later [Sec. 175]. Moreover, it has many advantages and therefore its preservation is fully justified.

The definition of the indefinite integral directly implies the following results.

1. $$d\int f(x)\,dx = f(x)\,dx,$$

i.e. the signs d and $\int$ *when the first precedes the second cancel each other.*

2. Since $F(x)$ is the primitive function for $F'(x)$ we have

$$\int F'(x)\,dx = F(x) + C,$$

which can be written in the form

$$\int dF(x) = F(x) + C.$$

We observe therefore that *the signs d and* $\int$, *before* $F(x)$, *cancel each other even when d follows* $\int$, *but then, however, we have to add an arbitrary constant to* $F(x)$.

Returning to the mechanical problem considered at the beginning of the section we may now write

$$v = \int a(t)\,dt \quad \text{and} \quad s = \int v(t)\,dt.$$

Suppose that, for the sake of definiteness, we are to deal with the uniformly accelerated motion, e.g. under the action of gravity; then $a = g$ (the downward direction of the vertical being considered positive) and, as can be easily understood

$$v = \int g\,dt = gt + C.$$

We have arrived at an expression for the velocity v which besides the time t also contains the arbitrary constant C. For various values of C we obtain various values for the velocity at the same instant of time; consequently, the data is as yet insufficient to solve the problem. To obtain a definite solution of the problem it is sufficient to know the velocity at any instant of time. For instance, suppose

that we know that at instant $t = t_0$ the velocity is $v = v_0$; substituting these values into the derived expression for the velocity, we find

$$v_0 = gt_0 + C, \quad \text{whence} \quad C = v_0 - gt_0.$$

Now our expression has a definite form,

$$v = g(t - t_0) + v_0.$$

Furthermore, we can find an expression for the distance s. We have

$$s = \int [g(t - t_0) + v_0]dt = \tfrac{1}{2}g(t - t_0)^2 + v_0(t - t_0) + C'$$

(it is easy to verify by differentiation that the primitive function can be taken in this form). The new unknown constant C' can be found if, for instance, we know that the distance $s = s_0$ at the instant $t = t_0$; then $C' = s_0$ and we can write the solution in the final form

$$s = \tfrac{1}{2}g(t - t_0)^2 + v_0(t - t_0) + s_0.$$

The values t_0, s_0, v_0 are called the *initial data* for the quantities t, s, v.

In exactly the same way we may write

$$m = \int \varrho(x)dx.$$

Here, again, a constant C appears in the integration; this is easily determined from the condition that for $x = 0$ the mass m vanishes.

156. The integral and the problem of determination of area. Since, historically, the concept of a primitive function has been very closely connected with the problem of the determination of areas, we shall consider this problem now (making use of the intuitive concept of the area of a plane figure and leaving the strict formulation to Chapter 12).

Consider in the interval $[a, b]$ a continuous function $f(x)$ taking positive (negative) values only. The figure $ABCD$ (Fig. 63) is bounded by the curve $y = f(x)$ and the ordinates $x = a$ and $x = b$ of the x-axis; this figure is called a *curvilinear trapezium*. To determine the area P of the figure we examine the behaviour of the area of the variable figure $AKLD$ contained between the initial ordinate $x = a$

and the ordinate corresponding to an arbitrarily selected value of x in the interval $[a, b]$. As x varies, the latter area varies accordingly and to every value of x there corresponds a definite value of the considered area; hence the area of the curvilinear trapezium $AKLD$ is a function of x, which we denote by $P(x)$.

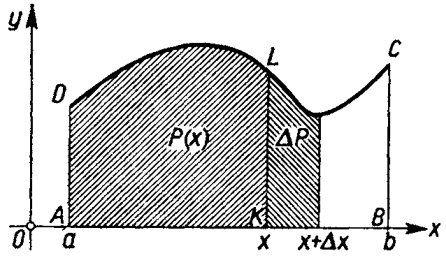

FIG. 63.

We first attempt to find the derivative of this function. Let x be given an increment Δx (positive, say); then the area $P(x)$ has an increment ΔP.

Denote by m and M, respectively, the smallest and the greatest values of the function $f(x)$ over the interval $[x, x+\Delta x]$ [Sec. 73], and let us compare the area ΔP with the areas of the rectangles constructed on the base Δx with heights m and M. Obviously,

$$m\,\Delta x < \Delta P < M\,\Delta x,$$

whence

$$m < \frac{\Delta P}{\Delta x} < M.$$

As $\Delta x \to 0$, m and M tend to $f(x)$ by continuity, and so

$$P'(x) = \lim_{\Delta x \to 0} \frac{\Delta P}{\Delta x} = f(x).$$

Thus we have arrived at a remarkable result, usually attributed to Newton and Leibniz†: *the derivative with respect to a finite abscissa of the variable area $P(x)$ is equal to the finite ordinate $y = f(x)$.*

In other words, the variable area $P(x)$ is the primitive function for the given function $y = f(x)$. Among the set of all primitive

† In fact, this proposition, in a different form, was published earlier by Isaac Barrow (1630–1677), Newton's teacher.

functions, this primitive function is distinguished by the property that it vanishes at $x = a$. Hence if we know some primitive $F(x)$ for $f(x)$, so that according to the theorem of the preceding section

$$P(x) = F(x) + C,$$

then we can easily determine the constant C; setting $x = a$ we have

$$0 = F(a) + C, \quad \text{so} \quad C = -F(a).$$

Thus, finally

$$P(x) = F(x) - F(a).$$

In particular, to derive the area P of the whole curvilinear trapezium $ABCD$ we set $x = b$:

$$P = F(b) - F(a).$$

As an example we find the area $P(x)$ of the figure bounded by a parabola $y = ax^2$, the ordinate corresponding to a given abscissa x and a segment of the x-axis

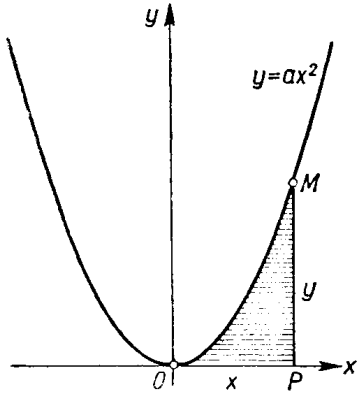

FIG. 64.

(Fig. 64). Since the parabola passes through the origin, $F(0) = 0$. It is easy to find the primitive function for $f(x) = ax^2$; it is $F(x) = ax^3/3$. This function vanishes at $x = 0$ and hence

$$P(x) = F(x) = \frac{ax^3}{3} = \frac{xy}{3}$$

[cf. Sec. 43, (3)].

In view of the connection between the evaluation of integrals and the determination of areas of plane figures, it became customary to also call the evaluation of the integrals squaring or quadrature.

To extend the above reasoning to functions which can also take negative values it is sufficient to agree to regard the areas of the parts of the figure located below the x-axis as *negative.*

Thus, for any function $f(x)$ continuous in the interval $[a, b]$ the reader may always represent the primitive function as a variable area bounded by the graph of the function. However, we obviously cannot regard this geometric illustration as a proof of the existence of the primitive function, since the concept of the area has not yet been justified.

In the following chapter [Sec. 183] we will be in a position to present a strict and purely analytic proof of the important fact that every function $f(x)$, continuous over an interval, has a primitive function in that interval. We anticipate this result and assume it to have been proved already.

In this chapter we only consider primitive functions for continuous functions. If the function is correctly prescribed and has points of discontinuity we consider it only over the intervals of its continuity. Therefore, assuming the validity of the above statement we avoid the necessity of assuming each time the existence of the integral: the integrals considered by us always exist.

157. Collection of the basic integrals. Every formula of the differential calculus establishing that the derivative of a function $F(x)$ is $f(x)$ leads directly to a corresponding formula of the integral calculus

$$\int f(x)\,dx = F(x) + C.$$

Examining the formulae of Sec. 81 by means of which the derivatives of elementary functions were computed, we are in a position to construct the following collection of integrals:

1. $\int 0 \cdot dx = C,$
2. $\int 1 \cdot dx = \int dx = x + C,$
3. $\int x^{\mu}\,dx = \frac{x^{\mu+1}}{\mu+1} + C \quad (\mu \neq -1),$
4. $\int \frac{1}{x}\,dx = \int \frac{dx}{x} = \log|x| + C,$

5. $\int \frac{1}{1+x^2}\,dx = \int \frac{dx}{1+x^2} = \arctan x + C,$

6. $\int \frac{1}{\sqrt{(1-x^2)}}\,dx = \int \frac{dx}{\sqrt{(1-x^2)}} = \arcsin x + C,$

7. $\int a^x\,dx = \frac{a^x}{\log a} + C, \quad \int e^x\,dx = e^x + C,$

8. $\int \sin x\,dx = -\cos x + C,$

9. $\int \cos x\,dx = \sin x + C,$

10. $\int \frac{1}{\sin^2 x}\,dx = \int \frac{dx}{\sin^2 x} = -\cot x + C,$

11. $\int \frac{1}{\cos^2 x}\,dx = \int \frac{dx}{\cos^2 x} = \tan x + C.$

Formula 4 requires an explanation. It can be applied in every interval which does not contain the origin. In fact, if this interval is located to the right of the origin, so that $x > 0$, then according to the familiar formula of differentiation $(\log x)' = 1/x$, we have at once

$$\int \frac{dx}{x} = \log x + C.$$

If the interval is located to the left of the origin and $x < 0$, differentiating, we easily find that $[\log(-x)]' = 1/x$; hence

$$\int \frac{dx}{x} = \log(-x) + C.$$

These two formulae are combined in formula 4.

The above collection of integrals can be extended by means of the following rules of integration.

158. Rules of integration. I. If a is a constant $(a \neq 0)$ then

$$\int a \cdot f(x)\,dx = a \cdot \int f(x)\,dx.$$

In fact, differentiating the expression on the right we have [Sec. 91, I]

$$d\left[a \cdot \int f(x)\,dx\right] = a \cdot d\left[\int f(x)\,dx\right] = a \cdot f(x)\,dx,$$

therefore this expression is the primitive function for the differential expression $a \cdot f(x)dx$, which was to be proved.

Thus, *a constant factor may be taken outside the integral sign.*

II. $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx.$

We differentiate the expression on the right [Sec. 91, II],

$$d\left[\int f(x)dx \pm \int g(x)dx\right] = d\int f(x)dx \pm d\int g(x)dx$$
$$= [f(x) \pm g(x)]dx;$$

this expression therefore is the primitive function for the last differential expression. This completes the proof.

The indefinite integral of a sum (difference) of differentials is equal to the sum (difference) of the integrals of each differential separately.

Remark. We make the following remark concerning the above formulae. They contain indefinite integrals, each containing an arbitrary term. Relations of this type are understood in the sense that the difference between the right- and left-hand sides is a constant. Alternatively we may interpret these relations literally, but then one of the integrals appearing is no longer an arbitrary primitive function; its constant is determined by the choice of the constants in the other integrals. This important remark should be remembered.

III. If

$$\int f(t)\,dt = F(t) + C,$$

then

$$\int f(ax + b)\,dx = \frac{1}{a} \cdot F(ax + b) + C'.$$

In fact, the above relation is equivalent to

$$\frac{d}{dt}F(t) = F'(t) = f(t).$$

But then

$$\frac{d}{dx}F(ax + b) = F'(ax + b) \cdot a = a \cdot f(ax + b),$$

and hence

$$\frac{d}{dx}\left[\frac{1}{a}F(ax+b)\right]=f(ax+b),$$

i.e. $(1/a)F(ax+b)$ is in fact the primitive function for $f(ax+b)$.

We frequently encounter the case $a=1$, $b=0$:

$$\int f(x+b)\,dx=F(x+b)+C_1, \quad \int f(ax)\,dx=\frac{1}{a}F(ax)+C_2.$$

(In fact, rule III is a particular case of the rule concerning the change of variable in an indefinite integral; we shall consider this problem in Sec. 160.)

159. Examples. *(1)* $\int (6x^2-3x+5)\,dx$.

Using rules II and I (and formulae 3, 2) we have

$$\begin{aligned}\int (6x^2-3x+5)\,dx &= \int 6x^2\,dx-\int 3x\,dx+\int 5\,dx\\ &= 6\int x^2\,dx-3\int x\,dx+5\int dx\\ &= 2x^3-\frac{3}{2}x^2+5x+C.\end{aligned}$$

A general polynomial can easily be integrated.

$$\begin{aligned}(2)\int (1+\sqrt{x})^4\,dx &= \int (1+4\sqrt{x}+6x+4x\sqrt{x}+x^2)\,dx\\ &= \int dx+4\int x^{\frac{1}{2}}\,dx+6\int x\,dx+4\int x^{\frac{3}{2}}\,dx+\int x^2\,dx\\ &= x+\frac{8}{3}x^{\frac{3}{2}}+3x^2+\frac{8}{5}x^{\frac{5}{2}}+\frac{1}{3}x^3+C. \qquad \text{(II, I; 3, 2)}\end{aligned}$$

$$\begin{aligned}(3)\int \frac{(x-\sqrt{x})(1+\sqrt{x})}{\sqrt[3]{x}}\,dx &= \int \frac{x\sqrt{x}-\sqrt{x}}{\sqrt[3]{x}}\,dx\\ &= \int x^{\frac{7}{6}}\,dx-\int x^{\frac{1}{6}}\,dx=\frac{6}{13}x^{\frac{13}{6}}-\frac{6}{7}x^{\frac{7}{6}}+C. \qquad \text{(II; 3)}\end{aligned}$$

We now give some examples on the application of rule III.

$$(4)\ (a)\int \frac{dx}{x-a}=\log|x-a|+C, \qquad \text{(III; 4)}$$

$$\begin{aligned}(b)\int \frac{dx}{(x-a)^k} &= \int (x-a)^{-k}\,dx\\ &\quad (k>1)\\ &= \frac{1}{-k+1}(x-a)^{-k+1}+C=-\frac{1}{(k-1)(x-a)^{k-1}}+C. \qquad \text{(III; 3)}\end{aligned}$$

(5) (a) $\int \sin mx\,dx = -\frac{1}{m}\cos mx + C,$ (III; 8)

$(m \neq 0)$

(b) $\int \cos mx\,dx = \frac{1}{m}\sin mx + C.$ (III; 9)

$(m \neq 0)$

(6) (a) $\int \frac{dx}{\sqrt{(a^2 - x^2)}} = \frac{1}{a}\int \frac{dx}{\sqrt{\left[1 - \left(\frac{x}{a}\right)^2\right]}} = \arcsin\frac{x}{a} + C,$ (III; 6)

$(a > 0)$

(b) $\int \frac{dx}{x^2 + a^2} = \frac{1}{a^2}\int \frac{dx}{1 + \left(\frac{x}{a}\right)^2} = \frac{1}{a}\arctan\frac{x}{a} + C.$ (III; 5)

Integration of a fraction with a complicated denominator can frequently be simplified by decomposing it into a sum of fractions with simpler denominators. For instance,

$$\frac{1}{x^2 - a^2} = \frac{1}{(x-a)(x+a)} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right);$$

and hence

(7) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\left[\int \frac{dx}{x-a} - \int \frac{dx}{x+a}\right] = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C.$

Some trigonometric expressions, after certain elementary transformations, can be integrated by means of simple methods.

For instance, obviously,

$$\cos^2 mx = \frac{1 + \cos 2mx}{2}, \quad \sin^2 mx = \frac{1 - \cos 2mx}{2},$$

hence

(8) (a) $\int \cos^2 mx\,dx = \frac{1}{2}x + \frac{1}{4m}\sin 2mx + C,$

$(m \neq 0)$

(b) $\int \sin^2 mx\,dx = \frac{1}{2}x - \frac{1}{4m}\sin 2mx + C.$

160. Integration by a change of variable. We now give a most effective method for the integration of functions—the method called the *change of variable* or the method of *substitution*. This is based on the following simple result.

If we know that

$$\int g(t)\,dt = G(t) + C,$$

then

$$\int g\big(\omega(x)\big)\omega'(x)\,dx = G\big(\omega(x)\big) + C.$$

(Each of the functions $g(t)$, $\omega(x)$, $\omega'(x)$ appearing here is assumed to be continuous.)

This follows directly from the rule of differentiation of a compound function [Sec. 84],

$$\frac{d}{dx}G\big(\omega(x)\big) = G'\big(\omega(x)\big)\omega'(x) = g\big(\omega(x)\big)\,\omega'(x),$$

if we bear in mind that $G'(t) = g(t)$. The same result can be expressed differently, since the relation

$$dG(t) = g(t)\,dt$$

remains valid when the independent variable t is replaced by a function $\omega(x)$ [Sec. 92].

Suppose that it is required to evaluate the integral

$$\int f(x)\,dx.$$

In many cases it is possible to select as the new variable a function of x, $t = \omega(x)$, such that the integral expression has the form

$$f(x)\,dx = g\big(\omega(x)\big)\omega'(x)\,dx, \tag{1}$$

where $g(t)$ is a function which is more easily integrated than $f(x)$. Then, by the above results, it is sufficient to find the integral

$$\int g(t)\,dt = G(t) + C,$$

and substituting $t = \omega(x)$ we obtain the required integral. Usually we simply write

$$\int f(x)\,dx = \int g(t)\,dt, \tag{2}$$

the substitution on the right-hand side being understood.

As an example we evaluate the integral

$$\int \sin^3 x \cos x\,dx.$$

Since $d\sin x = \cos x\,dx$, setting $t = \sin x$ we transform the integral expression into the form

$$\sin^3 x \cos x\,dx = \sin^3 x\,d\sin x = t^3\,dt.$$

The last integral is easily found:

$$\int t^3\,dt = \frac{t^4}{4} + C.$$

Returning to the variable x by replacing t by $\sin x$ we find

$$\int \sin^3 x \cos x\,dx = \frac{\sin^4 x}{4} + C.$$

We draw the reader's attention to the fact that when selecting the substitution $t = \omega(x)$ for the purpose of simplifying the integrand we have to remember that it must contain the factor $\omega'(x)\,dx$ representing the differential of the new variable (see (1)). In the preceding example the success of the substitution $t = \sin x$ was due to the presence of the factor $\cos x\,dx = dt$.

In this connection the following example is instructive:

$$\int \sin^3 x\,dx;$$

here the substitution $t = \sin x$ would be not applicable because the factor mentioned above is absent. If we attempt to separate out of the integrand, as the differential of the new variable the factor $\sin x\,dx$ or better $-\sin x\,dx$, we are led to the substitution $t = \cos x$; since the remaining expression

$$-\sin^2 x = \cos^2 x - 1$$

is simplified by this substitution, the latter is justified. We have

$$\int \sin^3 x\,dx = \int (t^2 - 1)\,dt = \frac{t^3}{3} - t + C = \frac{\cos^3 x}{3} - \cos x + C.$$

Sometimes the substitution is applied in a form different to that indicated above. We substitute a function $x = \varphi(t)$ of the new variable t directly into the integrand $f(x)\,dx$; this leads to the expression

$$f\big(\varphi(t)\big)\varphi'(t)\,dt = g(t)\,dt.$$

Evidently, if we now make the substitution $t = \omega(x)$ where $\omega(x)$ is the inverse function of $\varphi(t)$ we return to the original integrand $f(x)\,dx$. Hence, as before, relation (2) holds, where, on the right, after computing the integral we should set $t = \omega(x)$.

As an example, we compute the integral

$$\int \sqrt{(a^2 - x^2)}\, dx.$$

The difference of the squares under the root sign (the first being a constant) suggests the substitution $x = a \sin t$†. We have

$$\sqrt{(a^2 - x^2)} = a \cos t, \quad dx = a \cos t\, dt$$

and

$$\int \sqrt{(a^2 - x^2)}\, dx = a^2 \int \cos^2 t\, dt.$$

But we already know that the integral

$$a^2 \int \cos^2 t\, dt = a^2 \left[\frac{1}{2} t + \frac{1}{4} \sin 2t \right] + C$$

[Sec. 159, *(8)*]. To return to x we substitute $t = \arcsin (x/a)$; the transformation of the second term is simplified by the fact that

$$\frac{a^2}{4} \sin 2t = \frac{1}{2} a \sin t \cdot a \cos t = \frac{1}{2} x \sqrt{(a^2 - x^2)}.$$

Finally

$$\int \sqrt{(a^2 - x^2)}\, dx = \frac{1}{2} x \sqrt{(a^2 - x^2)} + \frac{a^2}{2} \arcsin \frac{x}{a} + C.$$

The ability to find convenient substitutions is developed by experience. Although we cannot give general rules for this, the reader will find certain particular remarks which facilitate this process in the next section. In the canonical cases the substitutions will simply be indicated in the text.

161. Examples. *(1)* (a) $\int e^{x^2} x\, dx$, (b) $\int \frac{x\, dx}{1 + x^4}$.

(a) Solution. Setting $t = x^2$ we have $dt = 2x\, dx$ and hence

$$\int e^{x^2} x\, dx = \frac{1}{2} \int e^t\, dt = \frac{1}{2} e^t + C = \frac{1}{2} e^{x^2} + C.$$

(b) Hint. The same substitution. Answer. $(1/2) \arctan x^2 + C$. In both cases the integrals have the form

$$\int g(x^2) x\, dx = \frac{1}{2} \int g(x^2)\, dx^2,$$

where g is an easily integrated function; for these integrals the substitution $t = x^2$ is natural.

(2) (a) $\int \frac{\log x}{x} dx$, (b) $\int \frac{dx}{x \log x}$, (c) $\int \frac{dx}{x \log^2 x}$.

---

† It should be observed that we assume that x ranges between $-a$ and a, while t between $-\pi/2$ and $\pi/2$. Consequently $t = \arcsin (x/a)$.

Hint. All these integrals have the form

$$\int g(\log x)\frac{dx}{x} = \int g(\log x)\,d\log x$$

and can be found by the substitution $t = \log x$.

Answer. (a) $\frac{1}{2}\log^2 x + C$; (b) $\log\log x + C$; (c) $-\frac{1}{\log x} + C$.

(3) Integrals of the form

$$\int g(\sin x)\cdot\cos x\,dx, \quad \int g(\cos x)\cdot\sin x\,dx, \quad \int g(\tan x)\cdot\frac{dx}{\cos^2 x}$$

are evaluated by means of the substitutions

$$t = \sin x, \quad t = \cos x, \quad t = \tan x,$$

respectively. For instance,

(a) $\int \frac{\cos x\,dx}{1+\sin^2 x} = \int \frac{dt}{1+t^2} = \arctan t + C = \arctan\sin x + C;$

(b) $\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx = -\int \frac{du}{u} = -\log|u| + C = -\log|\cos x| + C.$

(4) (a) $\int \frac{2x\,dx}{x^2+1}$, (b) $\int \cot x\,dx$.

Solution. (a) If we set $t = x^2 + 1$ the numerator $2x\,dx$ is dt and the integral is reduced to.

$$\int \frac{dt}{t} = \log|t| + C = \log(x^2+1) + C.$$

Observe that whenever the integral has the form

$$\int \frac{f'(x)}{f(x)}dx = \int \frac{df(x)}{f(x)},$$

the numerator of the integrand being the differential of the denominator, the substitution $t = f(x)$ reduces the integral immediately:

$$\int \frac{dt}{t} = \log|t| + C = \log|f(x)| + C.$$

Similarly we have

(b) $\int \cot x\,dx = \int \frac{d\sin x}{\sin x} = \log|\sin x| + C$ [cf. *(3)* (b)].

(5) $\int \frac{dx}{(x^2+a^2)^2}$.

The substitution is $x = a\tan t$†, $dx = \frac{a\,dt}{\cos^2 t}$, $x^2 + a^2 = \frac{a^2}{\cos^2 t}$.

† It is sufficient to assume that t varies between $-\pi/2$ and $\pi/2$.

Hence

$$\int \frac{dx}{(x^2+a^2)^2} = \frac{1}{a^3}\int \cos^2 t\, dt = \frac{1}{2a^3}(t+\sin t\cos t)+C \qquad \text{[cf. Sec 159, (8)]}.$$

We return to the variable x by setting $t = \arctan(x/a)$ and eliminate $\sin t$ and $\cos t$ by using $\tan t = x/a$. Finally

$$\int \frac{dx}{(x^2+a^2)^2} = \frac{1}{2a^2}\frac{x}{x^2+a^2}+\frac{1}{2a^3}\arctan\frac{x}{a}+C.$$

(6) $\displaystyle\int \frac{dx}{\sqrt{(x^2+\alpha)}}$ $(\alpha \gtrless 0)$.

Set $\sqrt{(x^2+\alpha)} = t-x$ and take t as the new variable. On squaring, x^2 may be omitted from both sides and we obtain

$$x = \frac{t^2-\alpha}{2t},$$

whence

$$\sqrt{(x^2+\alpha)} = t-\frac{t^2-\alpha}{2t} = \frac{t^2+\alpha}{2t}, \qquad dx = \frac{t^2+\alpha}{2t^2}dt.$$

Finally

$$\int \frac{dx}{\sqrt{(x^2+\alpha)}} = \int \frac{dt}{t} = \log|t|+C = \log|x+\sqrt{(x^2+\alpha)}|+C.$$

162. Integration by parts. Suppose that $u = f(x)$ and $v = g(x)$ are two functions of x which have continuous derivatives $u' = f'(x)$ and $v' = g'(x)$. Then by the rule of differentiation of a product, we have $d(uv) = u\,dv + v\,du$ or $u\,dv = d(uv) - v\,du$. It is evident that the primitive function for $d(uv)$ is uv; consequently we have the formula

$$\int u\,dv = uv - \int v\,du. \tag{3}$$

This formula expresses the *rule of integration by parts*. It reduces the integration of the expression $u\,dv = uv'\,dx$ to the integration of the expression $v\,du = vu'\,dx$.

For instance, suppose that we have to find the integral $\int x\cos x\,dx$. Set

$$u = x, \quad du = \cos x\,dx, \quad \text{whence} \quad du = dx, \quad v = \sin x\,\dagger$$

---

† Since it is sufficient for our purpose to represent $\cos x\,dx$ in any form dv there is no need to use the most general expression for v (i.e. that containing an arbitrary constant). This remark should henceforth be borne in mind.

and by (3)

$$\int x\cos x\,dx = \int x\,d\sin x = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C. \quad (4)$$

Thus, integration by parts makes it possible to replace the complicated integrand $x\cos x$ by the simple one $\sin x$. To obtain v we had to integrate the expression $\cos x\,dx$ (hence the name—integration by parts).

Applying formula (3) to the evaluation of the considered integral we have to split the integrand into two factors u and $dv = v'\,dx$, the first being differentiated while the second is integrated on passing to the integral on the right-hand side. One should try to proceed in such a way that the integration of the differential dv is easy and that the replacing of u by du and dv by v, as a whole, leads to a simplification of the integrand. Thus, in the example examined above it would certainly not be convenient to take, say, $x\,dx$ for dv and $\cos x$ for u.

On acquiring experience it becomes unnecessary to introduce u, v explicitly and we can apply the formula directly [cf. (4)].

The rule of integration by parts has a more restricted range of application than the change of the variable method. But there are classes of integrals, for instance,

$$\int x^k \log^m x\,dx, \quad \int x^k \sin bx\,dx, \quad \int x^k \cos bx\,dx, \quad \int x^k e^{ax}\,dx, \text{ etc.}$$

which are particularly amenable to the method of integrating by parts.

163. Examples. *(1)* $\int x^3 \log x\,dx$.

Differentiation of $\log x$ leads to a simplification of the integrand, so we set

$$u = \log x, \quad dv = x^3\,dx, \quad \text{whence} \quad du = \frac{dx}{x}, \quad v = \frac{1}{4}x^4$$

and thus

$$\int x^3 \log x\,dx = \frac{1}{4}x^4 \log x - \frac{1}{4}\int x^3\,dx = \frac{1}{4}x^4 \log x - \frac{1}{16}x^4 + C.$$

(2) (a) $\int \log x\,dx$, (b) $\int \operatorname{arc\,tan} x\,dx$.

Taking in both cases $dx = dv$ we obtain

(a) $\int \log x\,dx = x\log x - \int x\,d\log x = x\log x - \int dx = x(\log x - 1) + C$;

(b) $$\int \operatorname{arc\,tan} x\,dx = x \operatorname{arc\,tan} x - \int x\,d \operatorname{arc\,tan} x = x \operatorname{arc\,tan} x - \int \frac{x}{x^2+1}\,dx$$

$$= x \operatorname{arc\,tan} x - \frac{1}{2}\log(x^2+1) + C \qquad \text{[Sec. 161, } (4) \text{ (a)]}.$$

(3) $\int x^2 \sin x\,dx$.

We have

$$\int x^2\,d(-\cos x) = -x^2\cos x - \int (-\cos x)\,d(x^2)$$

$$= -x^2\cos x + 2\int x\cos x\,dx\,.$$

Thus we have reduced the required integral to a known one [Sec. 162, (4)]; substituting we obtain

$$\int x^2 \sin x\,dx = -x^2\cos x + 2(x\sin x + \cos x) + C.$$

Because the integrand was complicated we had to apply the rule of integration by parts twice.

Similarly, by the repeated application of this rule we can compute the integrals

$$\int P(x)e^{ax}\,dx, \qquad \int P(x)\sin bx\,dx, \qquad \int P(x)\cos bx\,dx,$$

where $P(x)$ is a polynomial in x.

(4) An interesting example is given by the integrals

$$\int e^{ax}\cos bx\,dx, \qquad \int e^{ax}\sin bx\,dx.$$

If we apply the method of integrating by parts (in both cases we set, say, $dv = e^{ax}\,dx$, $v = e^{ax}/a$) we obtain

$$\int e^{ax}\cos bx\,dx = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}\int e^{ax}\sin bx\,dx,$$

$$\int e^{ax}\sin bx\,dx = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\int e^{ax}\cos bx\,dx.$$

Thus, the integrals can be expressed in terms of each other†.

If we now substitute the expression for the second integral from the second formula into the first formula, we arrive at an equation for the first integral from which can be found:

$$\int e^{ax}\cos bx\,dx = \frac{b\sin bx + a\cos bx}{a^2+b^2}\,e^{ax} + C.$$

In an analogous way we find the second integral

$$\int e^{ax}\sin bx\,dx = \frac{a\sin bx - b\cos bx}{a^2+b^2}\,e^{ax} + C'.$$

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† If by integrals we mean definite primitive functions [see Sec. 158, Remark], then, wishing to have the same functions in the second formula as in the first, we should, strictly speaking, add a constant on the right. Of course, it would be contained in the constants C and C' in the final expressions.

(5) As a last example of the application of the method of integrating by parts we shall derive a *recurrence* formula for the evaluation of the integral

$$J_n = \int \frac{dx}{(x^2+a^2)^n} \qquad (n = 1, 2, 3, \ldots).$$

We apply formula (3) setting

$$u = \frac{1}{(x^2+a^2)^n}, \quad dv = dx, \quad \text{whence} \quad du = -\frac{2nx\cdot dx}{(x^2+a^2)^{n+1}}, \quad v = x.$$

We obtain

$$J_n = \frac{x}{(x^2+a^2)^n} + 2n\int \frac{x^2}{(x^2+a^2)^{n+1}}\,dx.$$

The last integral can be transformed as follows:

$$\int \frac{x^2}{(x^2+a^2)^{n+1}}\,dx = \int \frac{(x^2+a^2)-a^2}{(x^2+a^2)^{n+1}}\,dx$$

$$= \int \frac{dx}{(x^2+a^2)^n} - a^2\int \frac{dx}{(x^2+a^2)^{n+1}} = J_n - a^2 J_{n+1}.$$

Substituting this expression into the preceding relation we arrive at the relation

$$J_n = \frac{x}{(x^2+a^2)^n} + 2nJ_n - 2na^2 J_{n+1},$$

whence

$$J_{n+1} = \frac{1}{2na^2}\,\frac{x}{(x^2+a^2)^n} + \frac{2n-1}{2n}\,\frac{1}{a^2}\,J_n. \tag{5}$$

This formula reduces the computation of the integral J_{n+1} to that of the integral J_n where the index has been decreased by one. Knowing the integral

$$J_1 = \frac{1}{a}\arctan\frac{x}{a}$$

[Sec. 159, *(6)* (b): we take one of the values], taking $n = 1$ in formula (5) we find

$$J_2 = \frac{1}{2a^2}\,\frac{x}{x^2+a^2} + \frac{1}{2a^3}\arctan\frac{x}{a}$$

(this was already derived in another way—see Sec. 161, *(5)*). Setting $n = 2$ in formula (5) we obtain

$$J_3 = \frac{1}{4a^2}\,\frac{x}{(x^2+a^2)^2} + \frac{3}{4a^2}\,J_2 = \frac{1}{4a^2}\,\frac{x}{(x^2+a^2)^2} + \frac{3}{8a^4}\,\frac{x}{x^2+a^2} + \frac{3}{8a^5}\arctan\frac{x}{a}$$

and so on. Thus we can find the integral J_n for an arbitrary positive integral index.

§ 2. Integration of rational expressions

164. Formulation of the problem of integration in finite form. We have examined the elementary methods of computing indefinite integrals. These methods do not entirely determine the way to compute every integral but leave much to the reader's skill. In this and in the following sections we treat in more detail some particular but important classes of functions and we establish a definite procedure for their integration.

We begin by explaining what exactly we shall be considering, integrating functions of the above classes, and how these classes were selected.

In Sec. 25 we described the variety of functions to which analysis is first applied; these are the so-called elementary functions and functions which can be obtained from them by means of a finite number of arithmetical operations and compositions (without passing to a limit).

In Chapter 5 we found that all these functions are differentiable and their derivatives belong to the same class. The situation is different in the case of integrals; it often turns out that an integral of a function belonging to a certain class does not itself belong to that class, i.e. it cannot be expressed by elementary functions by means of a finite number of the kind of operations mentioned above. Among these integrals we have, for instance, the following:

$$\int e^{-x^2}\,dx, \quad \int \sin x^2\,dx, \quad \int \cos x^2\,dx,$$
$$\int \frac{\sin x}{x}\,dx, \quad \int \frac{\cos x}{x}\,dx, \quad \int \frac{dx}{\log x};$$

other similar examples will be given later [Secs. 169, 172 *et seq.*]. It is important to emphasize that all these integrals in fact exist†, but they represent entirely new functions and cannot be reduced to the functions which we have called "elementary".

Comparatively few classes of functions are known for which the integration can be performed in a finite form; these classes will be investigated here in detail. First of all we examine the class of rational functions.

† See the relevant text in Sec. 156. We shall return to this problem in Sec. 183.

165. Simple fractions and their integration. Since we can separate out from an improper rational fraction the integral part, the integration of which is easy, it is sufficient to investigate the integration of proper fractions (the degree of the numerator of which is lower than the degree of the denominator).

We consider here the so-called simple fractions; these are the fractions of the following four types:

$$\text{I.}\ \frac{A}{x-a},$$

$$\text{II.}\ \frac{A}{(x-a)^k}\ (k=2, 3, \ldots),$$

$$\text{III.}\ \frac{Mx+N}{x^2+px+q},$$

$$\text{IV.}\ \frac{Mx+N}{(x^2+px+q)^m}\ (m=2, 3, \ldots),$$

where A, M, N, a, p, q are real numbers; moreover, for the fractions of types III and IV we assume that the polynomial x^2+px+q has no real roots, i.e.

$$q-\frac{p^2}{4}>0.$$

The fractions of types I and II have already been integrated [Sec. 159, *(4)*], namely

$$A\int\frac{dx}{x-a}=A\log|x-a|+C,$$

$$A\int\frac{dx}{(x-a)^k}=-\frac{A}{k-1}\frac{1}{(x-a)^{k-1}}+C.$$

The integration of the fractions of types III and IV is facilitated by the following substitution. We separate from the expression x^2+px+q the square of the binomial

$$x^2+px+q=x^2+2\frac{p}{2}x+\left(\frac{p}{2}\right)^2+\left[q-\left(\frac{p}{2}\right)^2\right]$$

$$=\left(x+\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right).$$

The last expression in parenthesis is, by the above assumption, a positive number which we may denote by a^2, where we take

$$a = +\sqrt{\left(q - \frac{p^2}{4}\right)}.$$

We now substitute

$$x + \frac{p}{2} = t, \quad dx = dt,$$

$$x^2 + px + q = t^2 + a^2, \quad Mx + N = Mt + \left(N - \frac{Mp}{2}\right).$$

In case III we have

$$\int \frac{Mx+N}{x^2+px+q}\,dx = \int \frac{Mt + \left(N - \frac{Mp}{2}\right)}{t^2+a^2}\,dt$$

$$= \frac{M}{2}\int \frac{2t\,dt}{t^2+a^2} + \left(N - \frac{Mp}{2}\right)\int \frac{dt}{t^2+a^2}$$

$$= \frac{M}{2}\log(t^2+a^2) + \frac{1}{a}\left(N - \frac{Mp}{2}\right)\arctan\frac{t}{a} + C,$$

or returning to the variable x and substituting for a its value,

$$\int \frac{Mx+N}{x^2+px+q}\,dx$$

$$= \frac{M}{2}\log(x^2+px+q) + \frac{2N-Mp}{\sqrt{(4q-p^2)}}\arctan\frac{2x+p}{\sqrt{(4q-p^2)}} + C.$$

In type IV the same substitution yields

$$\int \frac{Mx+N}{(x^2+px+q)^m}\,dx = \int \frac{Mt + \left(N - \frac{Mp}{2}\right)}{(t^2+a^2)^m}\,dt$$

$$= \frac{M}{2}\int \frac{2t\,dt}{(t^2+a^2)^m} + \left(N - \frac{Mp}{2}\right)\int \frac{dt}{(t^2+a^2)^m}. \quad (1)$$

The first integral on the right can easily be computed by means of the substitution $t^2 + a^2 = u$, $2t\,dt = du$:

$$\int \frac{2t\,dt}{(t^2+a^2)^m} = \int \frac{du}{u^m} = -\frac{1}{m-1}\frac{1}{u^{m-1}} + C$$

$$= -\frac{1}{m-1}\frac{1}{(t^2+a^2)^{m-1}} + C. \qquad (2)$$

The second integral on the right, for an arbitrary m, can be computed by means of the recurrent formula [Sec. 163, *(5)*]. Then it only remains to set in the result $t = (2x+p)/2$ in order to return to the variable x.

This completes the problem of the integration of simple fractions.

166. Integration of proper fractions. Thus, we now know how to integrate simple fractions. The integration of an arbitrary proper fraction is based on the following important theorem which is proved in algebra.

Every proper fraction

$$\frac{P(x)}{Q(x)}$$

can be represented in the form of a sum of a finite number of simple fractions.

This decomposition of a proper fraction into simple fractions is connected with the resolution of the denominator into simple factors. It is known that every polynomial with real coefficients can be resolved (and moreover, uniquely) into real factors of the form $x-a$ and x^2+px+q; it is assumed here that the quadratic factors have no real roots and consequently they cannot be resolved into real linear factors. Collecting identical factors (if there are any) and assuming, for simplicity, that the highest coefficient of the polynomial $Q(x)$ is unity we can write the resolution of the polynomial in the form

$$Q(x) = \ldots\ (x-a)^k\ \ldots\ (x^2+px+q)^m \ldots, \qquad (3)$$

where $\ldots, k, \ldots, m, \ldots,$ are positive integers.

It should be observed that if the degree of the polynomial Q is n, then the sum of all the exponents k plus twice the sum of all the exponents m is n:

$$\sum k + 2\sum m = n. \tag{4}$$

It is established in algebra that to every factor of the form $(x-a)^k$ in the resolution of the denominator of the proper fraction, there corresponds a group of k simple fractions:

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \ldots + \frac{A_k}{(x-a)^k}, \tag{5}$$

and to every factor of the form $(x^2+px+q)^m$, a group of m simple fractions:

$$\frac{M_1x+N_1}{x^2+px+q} + \frac{M_2x+N_2}{(x^2+px+q)^2} + \ldots + \frac{M_mx+N_m}{(x^2+px+q)^m}, \tag{6}$$

A, M, N being numerical coefficients. Thus, knowing the resolution (3), we know the denominators of the simple fractions into which the considered fraction P/Q is resolved. Now consider the problem of determining the numerators, i.e. the coefficients A, M, N. Since the numerators of the group of fractions (5) contain k coefficients and the numerators of the group of fractions (6) $2m$ coefficients, then, by (4), there are altogether n coefficients.

To determine the required coefficients we usually use the method of undetermined coefficients, which is as follows. Knowing the form of the decomposition of the fraction P/Q we write it with literal coefficients in the numerators on the right. It is obvious that the common denominator of all simple fractions is Q; adding them, we obtain a proper fraction†. If we now omit the denominator Q on the left and on the right we arrive at an identity for two polynomials of the $(n-1)$th degree in x. The coefficients of the various powers of x of the polynomial on the right are linear homogeneous polynomials with respect to the n coefficients represented by the unknown letters; equating them to the corresponding numerical coefficients of the polynomial P, we finally obtain a system of n linear equations which enable us to find the unknown coefficients. Since the possi-

† A sum of proper rational fractions is always a proper fraction.

bility of the resolution into simple fractions has previously been established the derived system is never contradictory.

Furthermore, since the system of equations has a solution, its determinant is necessarily non-zero for any set of free terms (the coefficients of the polynomial P). In other words, the system can always be determined. This simple remark incidentally proves the uniqueness of the decomposition of a proper fraction into simple fractions.

We elucidate the above by an example.

Consider the fraction

$$\frac{2x^2+2x+13}{(x-2)(x^2+1)^2}.$$

By the above general theorem we have a decomposition

$$\frac{2x^2+2x+13}{(x-2)(x^2+1)^2}=\frac{A}{x-2}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}.$$

The coefficients A, B, C, D, E are determined from the identity

$$2x^2+2x+13=A(x^2+1)^2 + (Bx+C)(x^2+1)(x-2)+(Dx+E)(x-2).$$

Equating the coefficients of equal powers of x on the left and on the right, we arrive at a system of five equations

$$\begin{array}{c|l} x^4 & A+B=0, \\ x^3 & -2B+C=0, \\ x^2 & 2A+B-2C+D=2, \\ x^1 & -2B+C-2D+E=2, \\ x^0 & A-2C-2E=13, \end{array}$$

whence

$$A=1,\ B=-1,\ C=-2,\ D=-3,\ E=-4.$$

Finally

$$\frac{2x^2+2x+13}{(x-2)(x^2+1)^2}=\frac{1}{x-2}-\frac{x+2}{x^2+1}-\frac{3x+4}{(x^2+1)^2}.$$

The algebraic result which we have just established has a direct application to the integration of rational fractions. We found in the preceding section that the integrals of simple fractions were

elementary functions. Now we may state that the same is true for an arbitrary rational fraction. Considering the functions in terms of which the integrals of polynomials and proper fractions are expressed, we can formulate the following more precise result.

The integral of an arbitrary rational function can be expressed in a finite form in terms of rational functions, logarithms and inverse tangents.

For instance, returning to the above example and bearing in mind the formulae of Sec. 165 we have

$$\int \frac{2x^2+2x+13}{(x-2)(x^2+1)^2}\,dx = \int \frac{dx}{x-2} - \int \frac{x+2}{x^2+1}\,dx - \int \frac{3x+4}{(x^2+1)^2}\,dx$$

$$= \frac{1}{2}\,\frac{3-4x}{x^2+1} + \frac{1}{2}\log\frac{(x-2)^2}{x^2+1} - 4\arctan x + C.$$

Remark. The method of decomposition into simple fractions was originated by Leibniz. He had no difficulty in dealing with linear factors in the denominator, even with multiple roots. In the case of imaginary roots Leibniz compares each such root with its conjugate and from two imaginary linear expressions derives a real quadratic expression. However, he did not always succeed; thus he could not deduce the decomposition

$$x^4 + a^4 = (x^2 + \sqrt{2}ax + a^2)(x^2 - \sqrt{2}ax + a^2)$$

(this was later given by Taylor).

The determination of the numerators of simple fractions by means of the method of undetermined coefficients is due to Johann Bernoulli.

167. Ostrogradski's method for separating the rational part of an integral. Ostrogradski† discovered a method which greatly simplifies the evaluation of an integral of a rational proper fraction. This device enables one to separate out the rational part of the integral by a purely algebraic method.

We know [Sec. 165] that the rational terms of an integral appear, when integrating simple fractions, in the forms II and IV. In the first case the integral can be written down at once:

$$\int \frac{A}{(x-a)^k}\,dx = -\frac{A}{k-1}\,\frac{1}{(x-a)^{k-1}} + C. \tag{7}$$

We now proceed to establish the form of the rational part of the integral

$$\int \frac{Mx+N}{(x^2+px+q)^m}\,dx \qquad \left(m>1,\ q-\frac{p^2}{4}>0\right).$$

† Academician Mikhail Vasilyevitch Ostrogradski (1801–1861)—an outstanding Russian mathematician and specialist in mechanics.

Employing the familiar substitution $x+p/2=t$ we make use of relations (1), (2) and the reduction formula (5) of Sec. 163 for $n=m-1$. Returning to the variable x we obtain

$$\int \frac{Mx+N}{(x^2+px+q)^m}\,dx = \frac{M'x+N'}{(x^2+px+q)^{m-1}} + \alpha \int \frac{dx}{(x^2+px+q)^{m-1}},$$

where M', N', α denote certain constant coefficients. By means of the same formula, replacing m by $m-1$ we find for the last integral (if $m>2$)

$$\int \frac{\alpha\,dx}{(x^2+px+q)^{m-1}} = \frac{M''x+N''}{(x^2+px+q)^{m-2}} + \beta \int \frac{dx}{(x^2+px+q)^{m-2}},$$

etc., until the exponent of the trinomial in the integral on the right is unity. All the successively separated-out rational terms are proper fractions. Collecting them, we arrive at the following result:

$$\int \frac{Mx+N}{(x^2+px+q)^m}\,dx = \frac{R(x)}{(x^2+px+q)^{m-1}} + \lambda \int \frac{dx}{x^2+px+q}, \tag{8}$$

where $R(x)$ is an integral polynomial of a degree lower than the denominator† and λ is a constant.

Consider the proper fraction P/Q which is supposed to be irreducible and assume that its denominator Q is resolved into simple factors (see (3)). Then the integral of the fraction can be represented as the sum of integrals of fractions of the form (5) or (6). If k (or m) is greater than unity the integrals of all the fractions, other than the first, of the group (5) (or (6)) can be transformed by means of formula (7) (or (8)). Collecting all the results we finally arrive at a formula of the form

$$\int \frac{P(x)}{Q(x)}\,dx = \frac{P_1(x)}{Q_1(x)} + \int \frac{P_2(x)}{Q_2(x)}\,dx. \tag{9}$$

The rational part of the integral P_1/Q_1 is obtained by adding the above separated-out rational parts; consequently, it is a proper fraction and its denominator can be resolved as follows:

$$Q_1(x) = \ldots (x-a)^{k-1} \ldots (x^2+px+q)^{m-1} \ldots.$$

The fraction P_2/Q_2 which remained in the integrand was derived by adding fractions of the forms I and II, and therefore it is a proper fraction and

$$Q_2(x) = \ldots (x-a) \ldots (x^2+px+q) \ldots.$$

Obviously, (see (3)) $Q = Q_1Q_2$.

Formula (9) is called *Ostrogradski's formula.*

Differentiating, we obtain the equivalent form

$$\frac{P}{Q} = \left[\frac{P_1}{Q_1}\right]' + \frac{P_2}{Q_2}. \tag{10}$$

† See footnote on page. 322.

We know that the polynomials Q_1 and Q_2 can easily be found if the resolution (3) of the polynomial Q is known. In fact, since the derivative Q' contains all the simple fractions into which Q is resolved, but with exponents reduced by one, Q_1 is the greatest common divisor of Q and Q' and therefore can be determined in terms of these polynomials; this can be done, for instance, by the method of successive division. If Q_1 is known, Q_2 is found by simple division of Q by Q_1.

Now consider the determination of the numerators P_1 and P_2 in formula (10). For this purpose we also use the method of undetermined coefficients.

Denote the degrees of the polynomials Q, Q_1, Q_2 by n, n_1, n_2, respectively; hence $n_1 + n_2 = n$. Now the degrees of the polynomials P, P_1, P_2 are not higher than $n-1, n_1-1, n_2-1$. Now substitute for P_1 and P_2 polynomials of degrees n_1-1 and n_2-1 with unknown coefficients; altogether there are n_1+n_2, i.e. n coefficients. Differentiating (10):

$$\frac{P_1'Q_1 - P_1Q_1'}{Q_1^2} + \frac{P_2}{Q_2} = \frac{P}{Q}.$$

We now prove that the first fraction can always be reduced to the denominator Q, the numerator being a polynomial. We have

$$\frac{P_1'Q_1 - P_1Q_1'}{Q_1^2} = \frac{P_1'Q_2 - P_1\dfrac{Q_1'Q_2}{Q_1}}{Q_1Q_2} = \frac{P_1'Q_2 - P_1H}{Q},$$

where H denotes the quotient $Q_1'Q_2/Q_1$. But this quotient can be represented in the form of an entire polynomial. In fact, if Q_1 contains the factor $(x-a)^k$ for $k \geqslant 1$, then Q_1' contains the factor $(x-a)^{k-1}$ and Q_2 contains $(x-a)$; the same is true of the factor of the form $(x^2+px+q)^m$ for $m \geqslant 1$. Consequently, the numerator H is divisible by the denominator and henceforth by H we shall mean an entire polynomial (of degree n_2-1).

Eliminating the common denominator Q, we arrive at an identity containing the two polynomials (of degree $n-1$)

$$P_1'Q_2 - P_1H + P_2Q_1 = P.$$

Hence, as before, we obtain a system of n linear equations to determine the n unknown coefficients.

Since we have established the existence of the resolution (10) for any P, the above system of equations will hold identically in x. It follows therefore that its determinant is non-zero and consequently the system is necessarily solvable; thus the resolution (10), with the denominators Q_1 and Q_2, is unique†.

Example. It is required to separate out the rational part of the integral

$$\int \frac{4x^4 + 4x^3 + 16x^2 + 12x + 8}{(x+1)^2(x^2+1)^2}\,dx.$$

† See an analogous remark concerning the resolution of a proper fraction into simple fractions on p. 322.

We have

$$Q_1 = Q_2 = (x+1)(x^2+1) = x^3+x^2+x+1,$$

$$\frac{4x^4+4x^3+16x^2+12x+8}{(x^3+x^2+x+1)^2} = \left[\frac{ax^2+bx+c}{x^3+x^2+x+1}\right]' + \frac{dx^2+ex+f}{x^3+x^2+x+1},$$

whence

$$4x^4+4x^3+16x^2+12x+8 = (2ax+b)(x^3+x^2+x+1) - (ax^2+bx+c)(3x^2+2x+1) + (dx^2+ex+f)(x^3+x^2+x+1).$$

Equating the coefficients of equal powers on the two sides of the identity we arrive at a system of equations from which the unknowns $a, b, \ldots, f$ are determined:

x^5	$d = 0$ (henceforth we ignore d),	
x^4	$-a+e = 4$,	
x^3	$-2b+e+f = 4$,	$a = -1,\ b = 1,$
x^2	$a-b-3c+e+f = 16$,	$c = -4,\ d = 0,$
x^1	$2a-2c+e+f = 12$,	$e = 3,\quad f = 3.$
x^0	$b-c+f = 8.$	

Thus the required integral is

$$\int \frac{4x^4+4x^3+16x^2+12x+8}{(x+1)^2(x^2+1)^2}\,dx = -\frac{x^2-x+4}{x^3+x^2+x+1} + 3\int\frac{dx}{x^2+1} = -\frac{x^2-x+4}{x^3+x^2+x+1} + 3\arctan x + C.$$

In this example the computation of the last integral was easily performed directly. In other cases we may have to resolve into simple fractions again. Incidentally this stage of the calculation can be combined with the preceding one.

§ 3. Integration of some expressions containing roots

168. Integration of expressions of the form $R\left[x, \sqrt[m]{\left(\frac{\alpha x+\beta}{\gamma x+\delta}\right)}\right] dx$†.

We have previously examined the integration in a finite form of rational differentials. In what follows the basic method of integration of various classes of differential expressions is the determination of substitutions $t = \omega(x)$ (where ω itself is expressed in terms of elementary functions) which reduce the integrand to a rational form. We call this method the *method of rationalization of the integrand.*

† We henceforth denote rational functions by the letter R.

As a first example of its application consider the integral of the form

$$\int R\left[x, \sqrt[m]{\left(\frac{\alpha x+\beta}{\gamma x+\delta}\right)}\right] dx, \tag{1}$$

where R denotes a rational function of two arguments, m is a positive integer and $\alpha, \beta, \gamma, \delta$ are constants.

Set

$$t=\omega(x)=\sqrt[m]{\left(\frac{\alpha x+\beta}{\gamma x+\delta}\right)}, \quad t^m=\frac{\alpha x+\beta}{\gamma x+\delta}, \quad x=\varphi(t)=\frac{\delta t^m-\beta}{\alpha-\gamma t^m}.$$

Then we have for the integral

$$\int R[\varphi(t), t]\, \varphi'(t)\, dt,$$

where the differential already has a rational form, since R, φ, φ' are rational functions. Having evaluated this integral according to the rules of the preceding section we return to the original variable by substituting $t=\omega(x)$.

The following more general integrals can be reduced to an integral of the form (1):

$$\int R\left[x, \left(\frac{\alpha x+\beta}{\gamma x+\delta}\right)^r, \left(\frac{\alpha x+\beta}{\gamma x+\delta}\right)^s, \ldots\right] dx,$$

the exponents $r, s, \ldots$ being rational; it is sufficient to reduce these exponents to a common denominator m in order to obtain in the integrand a rational function of x and of the root

$$\sqrt[m]{\left(\frac{\alpha x+\beta}{\gamma x+\delta}\right)}.$$

Example.

$$\int \frac{dx}{\sqrt[3]{[(x-1)(x+1)^2]}}=\int \sqrt[3]{\left(\frac{x+1}{x-1}\right)} \frac{dx}{x+1}.$$

Setting

$$t=\sqrt[3]{\left(\frac{x+1}{x-1}\right)}, \quad x=\frac{t^3+1}{t^3-1}, \quad dx=-\frac{6t^2\, dt}{(t^3-1)^2},$$

we obtain

$$\int \sqrt[3]{\left(\frac{x+1}{x-1}\right)}\frac{dx}{x+1} = \int \frac{-3\,dt}{t^3-1} = \int\left(-\frac{1}{t-1}+\frac{t+2}{t^2+t+1}\right)dt$$

$$= \frac{1}{2}\log\frac{t^2+t+1}{(t-1)^2} + \sqrt{3}\arctan\frac{2t+1}{\sqrt{3}} + C,$$

where $t = \sqrt[3]{\left(\frac{x+1}{x-1}\right)}$.

169. Integration of binomial differentials. By the term *binomial differentials* we understand the following,

$$x^m(a+bx^n)^p\,dx,$$

where a, b are constants and the exponents m, n, p are rational numbers. We shall now establish the cases when these expressions can be integrated in a finite form.

One such case is at once clear: if p is an integer (positive, zero or negative) the above expression can be reduced to the form investigated in the preceding section. Namely, denoting by λ the smallest common multiple of the denominators of the fractions m and n we have an expression of the form $R(\sqrt[\lambda]{x})dx$ and hence the substitution $t = \sqrt[\lambda]{x}$ is sufficient for its rationalization.

We now transform the considered expression by means of the substitution $z = x^n$. Then

$$x^m(a+bx^n)^p dx = \frac{1}{n}(a+bz)^p z^{\frac{m+1}{n}-1}\,dz$$

and writing, for brevity

$$\frac{m+1}{n} - 1 = q,$$

we have

$$\int x^m(a+bx^n)^p\,dx = \frac{1}{n}\int (a+bz)^p z^q\,dz. \tag{2}$$

If q is an integer we again arrive at an expression of the type already investigated. In fact, denoting by ν the denominator of the fraction p, the transformed expression has the form $R[z, \sqrt[\nu]{(a+bz)}]$.

The rationalization of the integrand can also be achieved directly by means of the substitution

$$t = \sqrt[\nu]{(a+bz)} = \sqrt[\nu]{(a+bx^n)}.$$

Finally we can rewrite the second integral of (2) in the form

$$\int \left(\frac{a+bz}{z}\right)^p z^{p+q}\, dz.$$

It is readily observed that when $p+q$ is an integer we have a case already investigated: the integrand has the form $R\{z, \sqrt[\nu]{[(a+bz)/z]}\}$. The integrand in the integral considered can also be rationalized directly by means of the substitution

$$t = \sqrt[\nu]{\left(\frac{a+bz}{z}\right)} = \sqrt[\nu]{(ax^{-n}+b)}.$$

Thus, both the integrals of (2) can be expressed a finite form if one of the numbers

$$p, q, p+q$$

or, equivalently, one of the numbers

$$p, \frac{m+1}{n}, \frac{m+1}{n}+p$$

is an integer.

These cases of integrability were essentially known to Newton. However, it was not until the middle of the nineteenth century that Tchebychev established the remarkable result that no other binomial differentials are *integrable in a finite form.*

Let us examine some examples.

(1) $\int \frac{\sqrt[3]{(1+\sqrt[4]{x})}}{\sqrt{x}}\, dx = \int x^{-\frac{1}{2}}(1+x^{\frac{1}{4}})^{\frac{1}{3}}\, dx.$

Here $m = -1/2$, $n = 1/4$, $p = 1/3$; since

$$\frac{m+1}{n} = \frac{-(1/2)+1}{1/4} = 2,$$

we have the second case of integrability. Since $\nu = 3$, we set (in accordance with the general rule)

$$t = \sqrt[3]{(1+\sqrt[4]{x})}, \quad x = (t^3-1)^4, \quad dx = 12t^2(t^3-1)^3\, dt,$$

whence

$$\int \frac{\sqrt[3]{(1+\sqrt[4]{x})}}{\sqrt{x}}dx = 12\int (t^6 - t^3)dt = \frac{3}{7}t^4(4t^3 - 7) + C, \text{ etc.}$$

(2) $$\int \frac{dx}{\sqrt[4]{(1+x^4)}} = \int x^0(1+x^4)^{-\frac{1}{4}}dx.$$

Now $m = 0$, $n = 4$, $p = -1/4$ and we have an example of the third case of integrability, for $[(m+1)/n] + p = 1/4 - 1/4 = 0$. Now $\nu = 4$; setting

$$t = \sqrt[4]{(x^{-4}+1)} = \frac{\sqrt[4]{(1+x^4)}}{x}, \quad x = (t^4-1)^{-\frac{1}{4}},$$

$$dx = -t^3(t^4-1)^{-\frac{5}{4}}dt,$$

we have $\sqrt[4]{(1+x^4)} = tx = t(t^4-1)^{-1/4}$ and consequently

$$\int \frac{dx}{\sqrt[4]{(1+x^4)}} = -\int \frac{t^2\,dt}{t^4-1} = \frac{1}{4}\int\left(\frac{1}{t+1} - \frac{1}{t-1}\right)dt$$

$$-\frac{1}{2}\int \frac{dt}{t^2+1} = \frac{1}{4}\log\left|\frac{t+1}{t-1}\right| - \frac{1}{2}\arctan t + C,$$

etc.

170. Integration of expressions of the form $R[x, \sqrt{(ax^2+bx+c)}]$. Euler's substitution. We proceed to consider a very important class of integrals

$$\int R[x, \sqrt{(ax^2+bx+c)}]dx. \tag{3}$$

We assume that the quadratic trinomial does not have equal roots so that it cannot be replaced by a rational expression. We shall investigate the substitutions called *Euler's substitutions* by means of which we can always rationalize the integrand.

The first substitution is used in the case $a > 0$. Here we set

$$\sqrt{(ax^2+bx+c)} = t - \sqrt{a}\cdot x\dagger.$$

Squaring this relation, we find (subtracting the term ax^2 from both sides) $bx + c = t^2 - 2\sqrt{a}\cdot tx$. Hence

$$x = \frac{t^2-c}{2\sqrt{a}\cdot t + b}, \quad \sqrt{(ax^2+bx+c)} = \frac{\sqrt{a}\cdot t^2 + bt + c\sqrt{a}}{2\sqrt{a}\cdot t + b},$$

$$dx = 2\frac{\sqrt{a}\cdot t^2 + bt + c\sqrt{a}}{(2\sqrt{a}\cdot t + b)^2}\,dt.$$

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† We can also set $\sqrt{(ax^2+bx+c)} = t + \sqrt{a}\cdot x$.

The ingenuity of Euler's substitution lies in the fact that when determining x an equation of first degree is obtained, and hence x and also the root $\sqrt{(ax^2+bx+c)}$ can be expressed rationally in terms of t.

If the derived expression is substituted in (3) the problem reduces to the integration of a rational function of t. Consequently to return to x we have to set

$$t = \sqrt{(ax^2+bx+c)} + \sqrt{a}\cdot x.$$

The second substitution can be used if $c>0$. Now we set

$$\sqrt{(ax^2+bx+c)} = xt + \sqrt{c}^{\dagger}.$$

Squaring, subtracting c from both sides and dividing by x we obtain $ax+b = xt^2+2\sqrt{c}\cdot t$, i.e. again an expression of the first degree with respect to x. Therefore

$$x = \frac{2\sqrt{c}\cdot t - b}{a-t^2}, \quad \sqrt{(ax^2+bx+c)} = \frac{\sqrt{c}\cdot t^2 - bt + a\sqrt{c}}{a-t^2},$$

$$dx = 2\frac{\sqrt{c}\cdot t^2 - bt + \sqrt{c}\cdot a}{(a-t^2)^2}\,dt.$$

Obviously, substituting in (3) we have rationalized the integrand. Consequently, integrating we have

$$t = \frac{\sqrt{(ax^2+bx+c)} - \sqrt{c}}{x}.$$

Remark I. The cases considered above ($a>0, c>0$) can be reduced to each other with the help of the substitution $x = 1/z$. Therefore we can always avoid using the second substitution.

Finally, the third substitution is applicable in the case when the quadratic expression ax^2+bx+c has (distinct) real roots λ and μ. Then, as well known, this trinomial can be resolved into linear factors

$$ax^2+bx+c = a(x-\lambda)(x-\mu).$$

Setting

$$\sqrt{(ax^2+bx+c)} = t(x-\lambda),$$

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† Or $\sqrt{(ax^2+bx+c)} = xt - \sqrt{c}$.

squaring, and dividing by $x-\lambda$, we arrive at an equation of the first degree $a(x-\mu)=t^2(x-\lambda)$. Hence

$$x=\frac{-a\mu+\lambda t^2}{t^2-a}, \quad \sqrt{(ax^2+bx+c)}=\frac{a(\lambda-\mu)t}{t^2-a},$$

$$dx=\frac{2a(\mu-\lambda)t}{(t^2-a)^2}\,dt,$$

etc.

Remark II. Under our assumptions the root $\sqrt{[a(x-\lambda)(x-\mu)]}$ (assuming for definiteness, say, $x>\lambda$) can be transformed to the form

$$(x-\lambda)\sqrt{\left(a\frac{x-\mu}{x-\lambda}\right)},$$

and, consequently, in the case under consideration

$$R[x, \sqrt{(ax^2+bx+c)}]=R_1\left[x, \sqrt{\left(a\frac{x-\mu}{x-\lambda}\right)}\right].$$

Thus, essentially, we are faced with the differential investigated in Sec. 168. The third Euler's substitution which can be written in the form

$$t=\sqrt{\left(a\frac{x-\mu}{x-\lambda}\right)}$$

is identical with the substitution given in Sec. 168.

We now prove that the first and third Euler substitutions are, together, sufficient to carry out the rationalization of the integrand in (3) in all cases. In fact, if the trinomial ax^2+bx+c has real roots we know that the third substitution is applicable. If there are no real roots, i.e. $b^2-4ac<0$ the trinomial

$$ax^2+bx+c=\frac{1}{4a}[(2ax+b)^2+(4ac-b^2)]$$

has the same sign as a for all values of the variable x. The case $a<0$ is irrelevant since then the root has no real values. In the case $a>0$ the first substitution is applicable.

This reasoning also leads to the following general proposition: *integrals of the type* (3) *can always be evaluated in a finite form and can be represented by the integrals of rational differentials together with the square root sign.*

Examples. (1) We applied the first substitution in Sec. 161, *(6)* to the evaluation of the integral

$$\int \frac{dx}{\sqrt{(x^2 \pm a^2)}} \qquad (\alpha = \pm a^2).$$

Although the second basic integral

$$\int \frac{dx}{\sqrt{(a^2 - x^2)}}$$

is known from elementary considerations, for the sake of practice we evaluate it using Euler's substitutions.

(a) If we use the third substitution first,

$$\sqrt{(a^2 - x^2)} = t(a - x),$$

then

$$x = a\,\frac{t^2 - 1}{t^2 + 1}, \qquad dx = \frac{4at\,dt}{(t^2+1)^2}, \qquad \sqrt{(a^2 - x^2)} = \frac{2at}{t^2 + 1}$$

and

$$\int \frac{dx}{\sqrt{(a^2 - x^2)}} = 2\int \frac{dt}{t^2+1} = 2\arctan t + C = 2\arctan \sqrt{\left(\frac{a+x}{a-x}\right)} + C.$$

Using the identity

$$2\arctan \sqrt{\left(\frac{a+x}{a-x}\right)} = \arcsin\frac{x}{a} + \frac{\pi}{2} \qquad (-a < x < a),$$

we see that the result is distinct in form only from that already known to us.

The reader should henceforth always consider the possibility of the integral taking various forms, depending on the method of evaluation.

(b) If we apply the second substitution to the same integral

$$\sqrt{(a^2 - x^2)} = xt - a,$$

we obtain in an analogous way

$$\int \frac{dx}{\sqrt{(a^2 - x^2)}} = -2\int \frac{dt}{t^2+1} = -2\arctan t + C$$

$$= -2\arctan\frac{a + \sqrt{(a^2 - x^2)}}{x} + C.$$

We are faced here with an interesting phenomenon: the result holds separately for the intervals $(-a, 0)$ and $(0, a)$, since at the point $x = 0$ the expression

$$-2\arctan\frac{a + \sqrt{(a^2 - x^2)}}{x}$$

is meaningless. The limits as $x \to -0$ and $x \to +0$ of these expressions are distinct; they are equal to π and $-\pi$, respectively; choosing for the above intervals distinct

values of the constant C so that the second exceeds the first by 2π, we can construct a function which is continuous over the whole interval $(-a, a)$, if we take as its value at $x = 0$ the common limit from the left and from the right.

We have obtained the previous result in yet another form, as can be seen from the following identities:

$$-2\arctan\frac{a+\sqrt{(a^2-x^2)}}{x} = \begin{cases} \arcsin\dfrac{x}{a}-\pi & \text{for} \quad 0<x<a, \\ \arcsin\dfrac{x}{a}+\pi & \text{for} \quad -a<x<0. \end{cases}$$

(2) $\displaystyle\int \frac{dx}{x+\sqrt{(x^2-x+1)}}$.

(a) We first apply the first substitution $\sqrt{(x^2-x+1)} = t-x$, so

$$x = \frac{t^2-1}{2t-1}, \quad dx = 2\,\frac{t^2-t+1}{(2t-1)^2}\,dt,$$

$$\begin{aligned}\int \frac{dx}{x+\sqrt{(x^2-x+1)}} &= \int \frac{2t^2-2t+2}{t(2t-1)^2}\,dt \\ &= \int\left[\frac{2}{t}-\frac{3}{2t-1}+\frac{3}{(2t-1)^2}\right]dt \\ &= -\frac{3}{2}\cdot\frac{1}{2t-1}+2\log|t|-\frac{3}{2}\log|2t-1|+C.\end{aligned}$$

If we now substitute $t = x+\sqrt{(x^2-x+1)}$, we finally obtain

$$\begin{aligned}\int \frac{dx}{x+\sqrt{(x^2-x+1)}} &= -\frac{3}{2}\,\frac{1}{2x+2\sqrt{(x^2-x+1)}-1} \\ &\quad -\frac{3}{2}\log|2x+2\sqrt{(x^2-x+1)}-1|+2\log|x+\sqrt{(x^2-x+1)}|+C.\end{aligned}$$

(b) Applying first the second substitution $\sqrt{(x^2-x+1)} = tx-1$, we have:

$$x = \frac{2t-1}{t^2-1}, \quad dx = -2\frac{t^2-t+1}{(t^2-1)^2}\,dt, \quad \sqrt{(x^2-x+1)} = \frac{t^2-t+1}{t^2-1},$$

$$x+\sqrt{(x^2-x+1)} = \frac{t}{t-1},$$

$$\begin{aligned}\int \frac{dx}{x+\sqrt{(x^2-x+1)}} &= \int \frac{-2t^2+2t-2}{t(t-1)(t+1)^2}\,dt \\ &= \int\left[\frac{2}{t}-\frac{1}{2}\,\frac{1}{t-1}-\frac{3}{2}\,\frac{1}{t+1}-\frac{3}{(t+1)^2}\right]dt \\ &= \frac{3}{t+1}+2\log|t|-\frac{1}{2}\log|t-1|-\frac{3}{2}\log|t+1|+C'.\end{aligned}$$

It remains to substitute $t = [\sqrt{(x^2-x+1)}+1]/x$; after obvious simplifications we obtain

$$\int \frac{dx}{x+\sqrt{(x^2-x+1)}} = \frac{3x}{\sqrt{(x^2-x+1)}+x+1} + 2\log|\sqrt{(x^2-x+1)}+1| - \frac{1}{2}\log|\sqrt{(x^2-x+1)}-x+1| - \frac{3}{2}\log|\sqrt{(x^2-x+1)}+x+1| + C'.$$

This expression has a different form from the preceding one, but they are identical if we take $C' = C+3/2$.

§ 4. Integration of expressions containing trigonometric and exponential functions

171. Integration of the differentials $R(\sin x, \cos x)\,dx$. Differentials of this form can be rationalized by means of the substitution $t = \tan(x/2)\ (-\pi < x < \pi)$. In fact,

$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2t}{1+t^2}, \qquad \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{1-t^2}{1+t^2},$$

$$x = 2\arctan t, \qquad dx = \frac{2dt}{1+t^2},$$

whence

$$R(\sin x, \cos x)\,dx = R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\cdot\frac{2dt}{1+t^2}.$$

Thus integrals of the form

$$\int R(\sin x, \cos x)\,dx \tag{1}$$

can always be evaluated in a finite form; they can be represented in terms of the integrals of rational differentials together with the trigonometric functions.

The above substitution, which can always be used to solve integrals of type (1), sometimes leads to complicated calculations. We give some cases below which can be solved by means of simpler

substitutions. Prior to this we make the following remarks concerning some algebraic details.

If an integral or fractional rational function $R(u, v)$ remains the same when the sign of one of the arguments is changed (say, u), i.e. if

$$R(-u, v) = R(u, v),$$

it can be reduced to the form

$$R(u, v) = R_1(u^2, v),$$

containing only even powers of u.

If, however, on changing the sign of u the sign of $R(u, v)$ is reversed, i.e. if

$$R(-u, v) = -R(u, v),$$

then it can be reduced to the form

$$R(u, v) = R_2(u^2, v) \cdot u;$$

this follows directly from the preceding remark if we apply it to the function $R(u, v)/u$.

I. Suppose now that $R(u, v)$ changes its sign when the sign of u changes; then

$$\begin{aligned} R(\sin x, \cos x)\,dx &= R_0(\sin^2 x, \cos x)\sin x\,dx \\ &= -R_0(1-\cos^2 x, \cos x)\,d\cos x, \end{aligned}$$

and the rationalization is effected by the substitution $t = \cos x$.

II. Similarly, if $R(u, v)$ changes sign when the sign of v changes, we have

$$\begin{aligned} R(\sin x, \cos x)\,dx &= R_0^*(\sin x, \cos^2 x)\cos x\,dx \\ &= R_0^*(\sin x, 1-\sin^2 x)\,d\sin x, \end{aligned}$$

and now we use the substitution $t = \sin x$.

III. Suppose finally that the function $R(u, v)$ is unaltered by a simultaneous change in the signs of u and v:

$$R(-u, -v) = R(u, v).$$

In this case, replacing u by $(u/v)v$ we obtain

$$R(u, v) = R\left(\frac{u}{v}v, v\right) = R^*\left(\frac{u}{v}, v\right).$$

By the property of the function R, if we change the signs of u and v (so that the ratio u/v is unaltered)

$$R^*\left(\frac{u}{v}, -v\right) = R^*\left(\frac{u}{v}, v\right),$$

and so

$$R^*\left(\frac{u}{v}, v\right) = R_1^*\left(\frac{u}{v}, v^2\right).$$

Therefore

$$R(\sin x, \cos x) = R_1^*(\tan x, \cos^2 x) = R_1^*\left(\tan x, \frac{1}{1+\tan^2 x}\right),$$

i.e. simply

$$R(\sin x, \cos x) = \tilde{R}(\tan x).$$

Now substituting $t = \tan x$ $(-\pi/2 < x < \pi/2)$ we have

$$R(\sin x, \cos x)\,dx = \tilde{R}(t)\,\frac{dt}{1+t^2}, \text{ etc.}$$

Remark. Observe that, independently of its form, the rational expression $R(u, v)$ can always be written as the sum of three expressions of the particular types considered above. For instance, we can set

$$R(u, v) = \frac{R(u, v) - R(-u, v)}{2} + \frac{R(-u, v) - R(-u, -v)}{2} + \frac{R(-u, -v) + R(u, v)}{2}.$$

The first expression reverses its sign when the sign of u is changed, and the second when the sign of v is changed; the value of the third is unaltered in a simultaneous change of the signs of u and v. Resolving the expression $R(\sin x, \cos x)$ into the appropriate terms, we can apply to the first the substitution $t = \cos x$, to the second $t = \sin x$ and finally to the third the substitution $t = \tan x$. Thus, to evaluate integrals of type (1) these substitutions are sufficient.

Examples. (1) $\int \sin^2 x \cos^3 x\,dx$. The integrand changes its sign when $\cos x$ is replaced by $-\cos x$; thus we use the substitution $t = \sin x$, which yields

$$\int \sin^2 x \cos^3 x\,dx = \int t^2(1-t^2)\,dt = \frac{t^3}{3} - \frac{t^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.$$

(2) $\int \frac{dx}{\sin^4 x \cos^2 x}$. The sign of the integrand is unaltered if $\sin x$ is replaced by $-\sin x$ and $\cos x$ by $-\cos x$; thus we use the substitution $t = \tan x$, which yields

$$\int \frac{dx}{\sin^4 x \cos^2 x} = \int \frac{(1+t^2)^2}{t^4}\,dt = t - \frac{2}{t} - \frac{1}{3t^3} + C$$

$$= \tan x - 2\cot x - \frac{1}{3}\cot^3 x + C.$$

(3) $\int \frac{dx}{\sin x \cos 2x} = \int \frac{dx}{\sin x\,(2\cos^2 x - 1)}$. The substitution $t = \cos x$ yields

$$\int \frac{dx}{\sin x\,(2\cos^2 x - 1)} = \int \frac{dt}{(1-t^2)(1-2t^2)}$$

$$= \frac{1}{\sqrt{2}} \log \left| \frac{1+t\sqrt{2}}{1-t\sqrt{2}} \right| + \frac{1}{2} \log \left| \frac{1-t}{1+t} \right| + C$$

$$= \frac{1}{\sqrt{2}} \log \left| \frac{1+\sqrt{2}\cos x}{1-\sqrt{2}\cos x} \right| + \log \left| \tan \frac{x}{2} \right| + C.$$

(4) $\frac{1}{2} \int \frac{1-r^2}{1-2r\cos x + r^2}\,dx$ $(0 < r < 1,\ -\pi < x < \pi)$. We use here the generally applicable substitution $t = \tan(x/2)$. Thus we have

$$\frac{1}{2} \int \frac{1-r^2}{1-2r\cos x + r^2}\,dx = (1-r^2) \int \frac{dt}{(1-r)^2 + (1+r)^2 t^2}$$

$$= \arctan\left(\frac{1+r}{1-r}\,t\right) + C = \arctan\left(\frac{1+r}{1-r} \tan \frac{x}{2}\right) + C.$$

The integral

$$\int \frac{1-r\cos x}{1-2r\cos x + r^2}\,dx = \int \left[\frac{1}{2} + \frac{1}{2}\,\frac{1-r^2}{1-2r\cos x + r^2}\right] dx$$

$$= \frac{1}{2}\,x + \arctan\left(\frac{1+r}{1-r} \tan \frac{x}{2}\right) + C$$

can also be reduced to it.

172. Survey of other cases. In Sec. 163 we mentioned the method of integration for expressions of the form

$$P(x)\,e^{ax}\,dx, \quad P(x)\sin bx\,dx, \quad P(x)\cos bx\,dx,$$

where P is an integral polynomial. It is interesting to observe that the fractional expressions (n is a positive integer)

$$\frac{e^x}{x^n}\,dx, \quad \frac{\sin x}{x^n}\,dx, \quad \frac{\cos x}{x^n}\,dx$$

cannot be integrated in a finite form.

By integrating by parts we can easily establish reccurrence formulae for the integrals of these expressions and then reduce them to the three basic integrals

$$\text{I.}\quad \int \frac{e^x}{x}\,dx = \int \frac{dy}{\log y} = \operatorname{li} y^{\dagger} \qquad (\text{"integral logarithm"}),$$

$$\text{II.}\quad \int \frac{\sin x}{x}\,dx = \operatorname{si} x \qquad (\text{"integral sine"}),$$

$$\text{III.}\quad \int \frac{\cos x}{x}\,dx = \operatorname{ci} x \qquad (\text{"integral cosine"}^{\ddagger}).$$

We know [Sec. 163, *(4)*] that

$$\int e^{ax} \sin bx\,dx = \frac{a \sin bx - b \cos bx}{a^2 + b^2}\, e^{ax} + C,$$

$$\int e^{ax} \cos bx\,dx = \frac{b \sin bx + a \cos bx}{a^2 + b^2}\, e^{ax} + C.$$

From this we can explicitly evaluate the integrals

$$\int x^n e^{ax} \sin bx\,dx, \qquad \int x^n e^{ax} \cos bx\,dx,$$

where $n = 1, 2, 3, \ldots$. Thus, integrating by parts we obtain

$$\int x^n e^{ax} \sin bx\,dx = x^n \frac{a \sin bx - b \cos bx}{a^2 + b^2}\, e^{ax}$$

$$- \frac{na}{a^2 + b^2} \int x^{n-1} e^{ax} \sin bx\,dx + \frac{nb}{a^2 + b^2} \int x^{n-1} e^{ax} \cos bx\,dx,$$

$$\int x^n e^{ax} \cos bx\,dx = x^n \frac{b \sin bx + a \cos bx}{a^2 + b^2}\, e^{ax}$$

$$- \frac{nb}{a^2 + b^2} \int x^{n-1} e^{ax} \sin bx\,dx - \frac{na}{a^2 + b^2} \int x^{n-1} e^{ax} \cos bx\,dx.$$

These recurrence formulae make it possible to reduce the above integrals to the cases where $n = 0$.

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† The substitution is $x = \log y$.

‡ Incidentally, in all three cases it is necessary to fix the arbitrary constant; this will be done later.

§ 5. Elliptic integrals

173. Definitions. The integrals of the form

$$\int R[x, \sqrt{(ax^2+bx+c)}]\,dx$$

investigated in Sec. 170, which can be computed in a finite form, are naturally connected with integrals of the type

$$\int R[x, \sqrt{(ax^3+bx^2+cx+d)}]\,dx, \tag{1}$$

$$\int R[x, \sqrt{(ax^4+bx^3+cx^2+dx+e)}\,dx. \tag{2}$$

containing a square root of polynomials of the third or fourth degree. This is a very important class of integrals frequently encountered in applications. However, integrals of the form (1) and (2) cannot as a rule be expressed in a finite form in terms of elementary functions. Therefore, we have left their consideration to the last section, so as not to interrupt the basic course of exposition of the present chapter, which has been devoted mainly to the investigation of integrals expressible in a finite form.

The polynomials under the root are assumed to have real coefficients. Moreover, we assume that they have no multiple roots, for otherwise we could take a linear factor outside the root sign; then the problem would be reduced to integrating expressions of the types examined above and the integral could be expressed in a finite form. This may also occur when there are no multiple roots; for instance, it can easily be verified that

$$\int \frac{1+x^4}{1-x^4}\frac{dx}{\sqrt{(1-x^4)}} = \frac{x}{\sqrt{(1-x^4)}} + C,$$

$$\int \frac{5x^3+1}{\sqrt{(2x^3+1)}} = x\sqrt{(2x^3+1)} + C.$$

Integrals of the expressions of type (1) or (2) are generally called *elliptic* owing to the fact that they first appeared in solving the problem of the rectification of the ellipse [Sec. 201, *(4)*]. Incidentally this name, in the strict sense of the word, is applied only to integrals which are inexpressible in a finite form; the remaining integrals, of which the above are examples, are called pseudo-elliptic.

Investigation and construction of tables of values of the integrals of type (1) and (2) for arbitrary coefficients $a, b, c, \ldots$ is, of course, cumbersome. It is therefore natural to attempt to reduce all these integrals to a few which we hope would contain fewer arbitrary coefficient, (parameters).

174. Reduction to the canonical form. First of all we observe that it is generally sufficient to confine ourselves to the case of a polynomial of the fourth degree under the root, since the case of a polynomial of the third degree can easily be reduced to it. In fact, the polynomial of the third degree ax^3+bx^2+cx+d with real coefficients always has a real root [Sec. 69], say, λ, and consequently it can be resolved in the real form

$$ax^3+bx^2+cx+d = a(x-\lambda)(x^2+px+q).$$

The substitution $x-\lambda=t^2$ or $x-\lambda=-t^2$ produces the required reduction

$$\int R[x,\ \sqrt{(ax^3+\ldots)}]\,dx=\int R[t^2+\lambda,\ t\sqrt{(at^4+\ldots)}\,dt.$$

Thus, henceforth, we only consider integrals containing the root of a polynomial of the fourth degree, i.e. of the form (2).

By means of elementary transformations and substitutions, on which we cannot dwell here, every elliptic integral, as well as integrals expressible in a finite form, can be reduced to the so-called canonical form

$$\int\frac{R(z^2)\,dz}{\sqrt{[(1-z^2)(1-k^2z^2)]}},\tag{3}$$

where k is a positive proper fraction, $0<k<1$.

Separating from the rational function R the integral part and decomposing the remaining proper fraction into simple fractions we arrive at the following general proposition: all elliptic integrals can, by means of elementary substitutions (to within terms expressible in a finite form), be reduced to the following three standard integrals:

$$\int\frac{dz}{\sqrt{[(1-z^2)(1-k^2z^2)]}},\quad \int\frac{z^2dz}{\sqrt{[(1-z^2)(1-k^2z^2)]}}$$

and

$$(0<k<1)$$

$$\int\frac{dz}{(1+hz^2)\sqrt{[(1-z^2)(1-k^2z^2)]}};$$

h of the last integral may be complex. Liouville proved that these integrals cannot be expressed in a finite form. Legendre† called them elliptic integrals of the first, second and third kind, respectively. The first two contain only one parameter k while the last one also has the complex parameter h.

Legendre simplified these integrals further by performing the substitution $z=\sin\varphi$ (φ ranges from 0 to $\pi/2$). The first integral is then directly transformed into

$$\int\frac{d\varphi}{\sqrt{(1-k^2\sin^2\varphi)}},\tag{4}$$

which can be transformed as follows:

$$\int\frac{\sin^2\varphi\,d\varphi}{\sqrt{(1-k^2\sin^2\varphi)}}=\frac{1}{k^2}\int\frac{d\varphi}{\sqrt{(1-k^2\sin^2\varphi)}}-\frac{1}{k^2}\int\sqrt{(1-k^2\sin^2\varphi)}\,d\varphi,$$

i.e. it is reduced to the preceding integral together with the new integral

$$\int\sqrt{(1-k^2\sin^2\varphi)}\,d\varphi.\tag{5}$$

† Adrien Marie Legendre (1752–1833) and Joseph Liouville (1809–1882)—outstanding French mathematicians.

Finally, the above substitution transforms the third integral to the form

$$\int \frac{d\varphi}{(1+h\sin^2\varphi)\sqrt{(1-k^2\sin^2\varphi)}}. \tag{6}$$

Integrals (4), (5), (6) are also called elliptic integrals of the first, second and third kind in Legendre's form.

The first two are especially important and of frequent application. Assuming that these integrals vanish for $\varphi = 0$, thus determining the arbitrary constants, we obtain two definite functions of φ; Legendre denoted these by $F(k, \varphi)$ and $E(k, \varphi)$, respectively. Besides the independent variable φ the parameter k is indicated; it is called the *modulus*.

Legendre worked out extensive tables of values of these functions for various φ and k. The argument φ regarded as an angle was expressed in degrees, and moreover the modulus k (a proper fraction) was regarded as the sine of an angle θ which was also expressed in degrees.

Furthermore, Legendre and other scholars investigated the basic properties of these functions and established numerous formulae relating to them. Owing to this the functions F and E of Legendre were introduced into the set of functions used in analysis and its applications on equal terms with the elementary functions.

The first part of the integral calculus to which we have essentially confined ourselves deals with "integration in a finite form". It would, however, be erroneous to assume that the theory of integral calculus is confined to this; the elliptic integrals F and E are examples of functions which can be investigated successfully from their definition as integral expressions and can be usefully applied in this form even though they cannot be expressed in terms of elementary functions in a finite form.

CHAPTER 11

DEFINITE INTEGRAL

§ 1. Definition and conditions for the existence of a definite integral

175. Another formulation of the area problem. We return to the problem of determining the area P of the curvilinear trapezium $ABCD$ (Fig. 65) which was considered in Sec. 156. We now present a different formulation of the solution of this problem†.

We divide the base AB of the figure into parts in an arbitrary way and construct the ordinates corresponding to the points of division; then the curvilinear trapezium is divided into a number of strips (see Fig. 65).

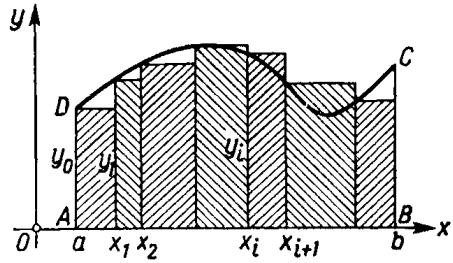

FIG. 65.

We now replace approximately every strip by a rectangle the base of which is the same as that of the strip and the height of which is equal to one of the ordinates of the strip, the left-hand one, say. In this way the curvilinear trapezium is replaced by a step figure composed of rectangles.

The abscissae of the points of division are denoted as follows:

$$a = x_0 < x_1 < x_2 < \ldots < x_i < x_{i+1} < \ldots < x_n = b. \qquad (1)$$

The base of the rectangle $(i = 0, 1, \ldots, n-1)$ is evidently equal

† Generalizing the idea applied already in the particular example [Sec. 43 *(3)*].

to the difference, $x_{i+1}-x_i$, which we denote by Δx_i. According to the above assumptions the height is equal to $y_i = f(x_i)$. Consequently the area of the ith rectangle is $y_i \Delta x_i = f(x_i)\Delta x_i$.

Summing the areas of all the rectangles we obtain an approximate value of the area P of the curvilinear trapezium:

$$P \doteq \sum_{i=0}^{n-1} y_i \Delta x_i \quad \text{or} \quad P \doteq \sum_{i=0}^{n-1} f(x_i)\Delta x_i.$$

The error of this relation when all Δx_i become infinitely small tends to zero. The exact value of the area P is obtained as the limit

$$P = \lim \sum y_i \Delta x_i = \lim \sum f(x_i)\Delta x_i, \tag{2}$$

assuming that all the lengths Δx_i simultaneously tend to zero.

The same method is also applicable to the computation of the area $P(x)$ of the figure $AKLD$ (Fig. 63); now the segment AK only has to be subdivided. Observe that the case when $y = f(x)$ may also take negative values is solved by the condition of Sec. 156 that the areas of the parts of the figure under the x-axis are negative.

To denote the sum of the form $\sum y \Delta x$ (or, strictly speaking, its limiting value) Leibniz introduced the symbol $\int y\,dx$ where $y\,dx$ resembles the typical term of the sum and $\int$ is an elongated S—the first letter of the Latin word *summa*†. Since the area representing this limiting value is also the primitive function for $f(x)$ the same symbol has been used as for the primitive function. Subsequently, after introducing the functional notation, the symbol

$$\int f(x)\,dx$$

was used for a variable area, while we wrote

$$\int_a^b f(x)\,dx$$

for the area of a figure $ABCD$ lying between the abscissae $x = a$ and $x = b$.

Above we have made use of the intuitive idea of area in order to tackle the limits of the various sums of the form (2) in a natural way, following the historical development of this problem. However, the very concept of area requires justification and, when speaking above of a curvilinear trapezium, it depended on the existence of the

† The term "integral" (from the Latin *integer*—entire) was proposed by the disciple and associate of Leibniz—John Bernoulli. Leibniz himself originally used the expression "the sum of all $y\,dx$".

above limits. Evidently, the limits (2) should be investigated independently of the geometric representation of the function $f(x)$; the present chapter is devoted to this task.

Limits of the type (2) play an important part in mathematical analysis and its various applications. Moreover, various modifications of these concepts will frequently be encountered in this course of analysis.

176. Definition. Suppose that the function $f(x)$ is given over an interval $[a, b]$. We subdivide this interval arbitrarily by introducing between a and b the points of division (1). The greatest of the differences $\Delta x_i = x_{i+1} - x_i (i = 0, 1, \ldots, n-1)$ will henceforth be denoted by λ.

Take some arbitrary point $x = \xi_i$† in every subinterval $[x_i, x_{i+1}]$

$$x_i \leqslant \xi_i \leqslant x_{i+1} \quad (i = 0, 1, \ldots, n-1)$$

and form the sum

$$\sigma = \sum_{i=0}^{n-1} f(\xi_i)\, \Delta x_i.$$

We now proceed to establish the existence of a (finite) limit of this sum

$$I = \lim_{\lambda \to 0} \sigma. \tag{3}$$

Suppose that the interval $[a, b]$ is successively divided into parts, first in one way, then in another, and so on. This sequence of divisions of the interval into parts will be called fundamental if the corresponding sequence of values $\lambda = \lambda_1, \lambda_2, \lambda_3, \ldots$ tends to zero.

Relation (3) is understood in the sense that the sequence of values of the sum σ corresponding to an arbitrary fundamental sequence of divisions of the interval always tends to a limit I for all possible values of ξ_i.

Here also the limit can be defined in the "language ε–δ". Namely, it is claimed that the sum σ for $\lambda \to 0$ has the limit I, if, for any number $\varepsilon > 0$, a number $\delta > 0$ can be found, such that provided $\lambda < \delta$ (i.e. the fundamental interval is divided into parts with lengths $\Delta x_i < \delta$) the inequality

$$|\sigma - I| < \varepsilon$$

is valid however the numbers ξ_i are chosen.

† Previously ξ_i was taken to be the smallest value of the subinterval.

The proof of the equivalence of the two definitions can be carried out similarly to the proof in Sec. 33. The first definition in the "language of sequences" makes it possible to transfer the basic concepts and theorems of the theory of limits to the new form of the limit.

The finite limit I of the sum σ when $\lambda \to 0$ is called the definite integral of the function $f(x)$ in the interval from a to b, and it is denoted by the symbol†

$$I = \int_a^b f(x)\,dx; \tag{4}$$

then the function $f(x)$ is said to be integrable over the interval $[a, b]$.

The numbers a and b are called the *lower and upper limits* of the integral. A definite integral with constant limits represents a constant number.

The foregoing definition was given by Riemann, who first announced it in the general form and investigated its domain of application. The sum σ itself is sometimes called Riemann's sum, although Cauchy had earlier used limits of similar sums for the case of continuous functions. We prefer here to call it the integral sum to emphasize its connection with the integral.

We now attempt to find conditions under which the integral sum σ has a finite limit, i.e. the definite integral (4) exists.

First of all it should be observed that the definition stated may in fact be applied to a bounded function only. In fact, if function $f(x)$ were unbounded in the interval $[a, b]$, then for any subdivision of the interval the function would be unbounded in at least one of the subintervals. Then by an appropriate selection of the point ξ in this subinterval we could make $f(\xi)$, and consequently the sum σ, arbitrarily large; it is therefore clear that σ could have no limit. Thus, an integrable function is necessarily bounded.

Consequently in what follows we shall assume *a priori* that the function $f(x)$ is bounded

$$m \leqslant f(x) \leqslant M, \quad \text{if} \quad a \leqslant x \leqslant b.$$

---

† This notation was introduced by the French mathematician and physicist Jean Baptiste Joseph Fourier (1768–1830). Euler used a more cumbersome notation:

$$\int P\,dx \begin{bmatrix} \text{from} & x = a \\ \text{to} & x = b \end{bmatrix}.$$

177. Darboux's sums. To facilitate the investigation, following Darboux† we introduce, besides the integral sums, other similar but simpler sums.

Denote by m_i and M_i the exact lower and upper bounds of the function $f(x)$ in the ith interval $[x_i, x_{i+1}]$ and form the sum

$$s = \sum_{i=0}^{n-1} m_i \Delta x_i, \qquad S = \sum_{i=0}^{n-1} M_i \Delta x_i.$$

These sums are called the lower and upper (integral) sums, respectively (or Darboux's sums).

In the particular case when $f(x)$ is continuous they are simply the smallest and the greatest of the integral sums corresponding to the given subdivision, since in this case the function $f(x)$ attains its exact bounds in every subinterval and the points ξ_i can be selected in such a way that

$$f(\xi_i) = m_i \quad \text{or} \quad f(\xi_i) = M_i.$$

Proceeding to the general case we have, from the definition of the lower and upper bounds,

$$m_i \leqslant f(\xi)_i \leqslant M_i.$$

Multiplying Δx_i (Δx_i is positive) and summing with respect to i we obtain

$$s \leqslant \sigma \leqslant S.$$

For a fixed division the sums s and S are constant while the sum σ can vary in view of the arbitrariness of the numbers ξ_i. It is readily observed, however, that by a suitable choice of ξ_i the values $f(\xi_i)$ can be made arbitrarily near either m_i or M_i and consequently the sum σ can be made arbitrarily near s or S. Then the above inequalities imply the following general remark. *For a given division of the interval, the Darboux sums s and S are respectively the greatest lower and least upper bounds of the integral sums.*

The Darboux sums possess the following simple properties.

FIRST PROPERTY. *If any set of points subdividing $[a, b]$ is augmented by further points within the interval the lower Darboux sum can only increase while the upper sum can only decrease.*

† Gaston Darboux (1842–1917)—a French mathematician.

Proof. To prove this property it is sufficient to examine the effect of the addition of just one further point of subdivision, x', say.

Suppose that this point lies between points x_k and x_{k+1}, so

$$x_k < x' < x_{k+1}.$$

Denoting by S' the new upper sum, we observe that it differs from the former one only in the fact that in S the interval $[x_k, x_{k+1}]$ was associated with the term

$$M_k(x_{k+1} - x_k),$$

while in the new sum S' this interval is associated with the sum of two terms

$$\overline{M}_k(x' - x_k) + \overline{\overline{M}}_k(x_{k+1} - x'),$$

where $\overline{M}_k$ and $\overline{\overline{M}}_k$ are the least upper bounds of the function $f(x)$ in the intervals $[x_k, x']$ and $[x', x_{k+1}]$. Since these intervals are parts of the interval $[x_k, x_{k+1}]$ we have

$$\overline{M}_k \leqslant M, \quad \overline{\overline{M}}_k \leqslant M_k,$$

whence

$$\overline{M}_k(x' - x_k) \leqslant M_k(x' - x_k), \quad \overline{\overline{M}}_k(x_{k+1} - x') \leqslant M_k(x_{k+1} - x').$$

Adding these inequalities we obtain

$$\overline{M}_k(x' - x_k) + \overline{\overline{M}}_k(x_{k+1} - x') \leqslant M_k(x_{k+1} - x_k).$$

This implies that $S' \leqslant S$. The proof for the lower sum is analogous.

SECOND PROPERTY. *Any lower Darboux sum is less than every upper sum, even if the latter corresponds to another subdivision of the interval.*

Proof. Divide the interval $[a, b]$ in an arbitrary way into parts and form for this subdivision the Darboux sums

$$s_1 \quad \text{and} \quad S_1. \tag{I}$$

Now consider another subdivision of the interval $[a, b]$, unrelated to the first one. The appropriate Darboux sums are

$$s_2 \quad \text{and} \quad S_2. \tag{II}$$

It is required to prove that $s_1 \leqslant S_2$. For this purpose we combine the two sets of points corresponding to the two methods of sub-

division; then we obtain a third subdivision with the associated sums

$$s_3 \quad \text{and} \quad S_3. \tag{III}$$

The third subdivision has been obtained from the first by adding new points to it; therefore by the First Property of Darboux sums we have

$$s_1 \leqslant s_3.$$

Now comparing the second and the third subdivisions in exactly the same way we see that

$$S_3 \leqslant S_2.$$

But $s_3 \leqslant S_3$ and consequently it follows from the above inequalities that

$$s_1 \leqslant S_2.$$

This completes the proof.

It follows from the above proof that the whole set $\{s\}$ of the lower sums is bounded above by any upper sum S. Therefore [Sec. 6], this set has a finite least upper bound

$$I_* = \sup \{s\}$$

and moreover

$$I_* \leqslant S,$$

for any upper sum S. Therefore since the set of upper sums is bounded below by the number I_*, it has a finite greatest lower bound

$$I^* = \inf \{S\},$$

and evidently

$$I_* \leqslant I^*.$$

We infer from the above results that

$$s \leqslant I_* \leqslant I^* \leqslant S \tag{5}$$

for arbitrary lower and upper Darboux sums.

178. Condition for the existence of the integral. Such a condition can easily be formulated in terms of the Darboux sums.

THEOREM. *For the existence of a definite integral it is necessary and sufficient that*

$$\lim_{\lambda \to 0} (S - s) = 0. \tag{6}$$

The results of Sec. 176 are sufficient to establish the sense of this limit. For instance, in the "ε–δ language", condition (6) means that for any $\varepsilon > 0$ a number $\delta > 0$ can be found, such that, provided $\lambda < \delta$ (i.e. the interval is divided into parts the lengths of which $\Delta x_i < \delta$), the inequality

$$S - s < \varepsilon$$

is satisfied.

Proof. Necessity. Assume that the integral (4) exists. Then for any given $\varepsilon > 0$ a number $\delta > 0$ can be found such that, provided $\Delta x_i < \delta$, we have

$$|\sigma - I| < \varepsilon \quad \text{or} \quad I - \varepsilon < \sigma < I + \varepsilon,$$

independently of the choice of ξ_i within the bounds of the corresponding subintervals. But for the given subdivision of the interval the sums s and S are the greatest lower and least upper bounds, respectively, of the integral sums; consequently we have

$$I - \varepsilon \leqslant s \leqslant S \leqslant I + \varepsilon. \tag{7}$$

Hence

$$\lim_{\lambda \to 0} s = I, \quad \lim_{\lambda \to 0} S = I,$$

which implies (6).

Sufficiency. Assume that the condition (6) is satisfied; then it is clear from (5) that $I_* = I^*$ and, denoting their common value by I, we have

$$s \leqslant I \leqslant S. \tag{5*}$$

If by σ we understand one of the values of the integral sum corresponding to the same subdivision of the interval as defines the sums s and S, then we know that

$$s \leqslant \sigma \leqslant S.$$

By condition (6), if we assume that all the Δx_i are sufficiently small, the sums s and S will differ by less than an arbitrarily chosen $\varepsilon > 0$. Therefore the same will be true for the numbers σ and I lying between s and S, i.e.

$$|\sigma - I| < \varepsilon.$$

Consequently I is the limit of σ, i.e. it is the definite integral.

Denoting the oscillation $M_i - m_i$ of the function in the ith subinterval by ω_i, we have

$$S - s = \sum_{i=0}^{n-1} (M_i - m_i)\Delta x_i = \sum_{i=0}^{n-1} \omega_i \Delta x_i,$$

and the condition for the existence of the definite integral can be written in the form

$$\lim_{\lambda \to 0} \sum_{i=0}^{n-1} \omega_i \Delta x_i = 0. \tag{8}$$

This is the customary form.

179. Classes of integrable functions. We now apply the criterion deduced above to establish some classes of integrable functions.

I. *If the function $f(x)$ is continuous over the interval $[a, b]$, it is integrable.*

Proof. Since the function $f(x)$ is continuous, by the corollary to Cantor's theorem [Sec. 75], given $\varepsilon > 0$ a number $\delta > 0$ can always be found such that, provided the interval $[a, b]$ is subdivided into parts of lengths $\Delta x_i < \delta$, we always have $\omega_i < \varepsilon$. Hence

$$\sum_{i=0}^{n-1} \omega_i \Delta x_i < \varepsilon \sum_{i=0}^{n-1} \Delta x_i = \varepsilon(b - a).$$

Since $b - a$ is a constant number and ε is arbitrarily small, condition (8) is satisfied and it implies the existence of the integral. The above statement can be somewhat generalized.

II. *If the function $f(x)$, bounded in $[a, b]$, has a finite number of points of discontinuity only, then it is integrable.*

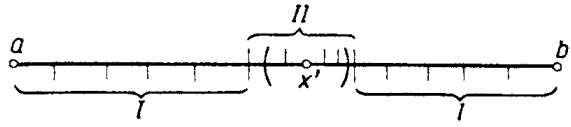

FIG. 66.

Proof. We confine ourselves to the case when there is only one point of discontinuity x' between a and b (Fig. 66). Take an arbitrary $\varepsilon > 0$ and surround the point x' by the interval $(x' - \varepsilon, x' + \varepsilon)$. In the two remaining (closed) intervals the function $f(x)$ is continuous

and we may apply the corollary of Cantor's theorem to each of them separately. Let δ be the lesser of the two numbers defined by the corollary [see Sec. 75]; then this δ satisfies the conditions of the corollary over $[a, x'-\varepsilon]$ and $[x'+\varepsilon, b]$; moreover, without loss of generality we may take $\delta < \varepsilon$. We now arbitrarily subdivide the interval $[a, b]$ into parts the lengths Δx_i of which are all smaller than δ. The resulting subintervals are of two kinds:

(I) intervals which wholly lie outside the separated neighbourhood of the point of discontinuity; in these intervals the oscillation of the function $\omega_i < \varepsilon$;

(II) intervals lying either wholly or partly inside the above neighbourhood of the discontinuity.

Since we assumed that the function $f(x)$ is bounded, its oscillation in any of these intervals does not exceed the oscillation Ω in the whole interval $[a, b]$.

The sum

$$\sum_i \omega_i \Delta x_i$$

is now divided into two sums

$$\sum_{i'} \omega_{i'} \Delta x_{i'} \quad \text{and} \quad \sum_{i''} \omega_{i''} \Delta x_{i''},$$

extended over the intervals of the first and second kinds, respectively.

As in the preceding theorem, we have for the first sum

$$\sum_{i'} \omega_{i'} \Delta x_{i'} < \varepsilon \sum_{i'} \Delta x_{i'} < \varepsilon(b-a).$$

For the second sum we observe that the sum of the lengths of the intervals of the second kind lying wholly inside the separated neighbourhood is less than or equal to 2ε; now, there can be no more than two subintervals only partly covering the considered neighbourhood and the sum of their lengths is smaller than 2δ, and therefore smaller than 2ε. Consequently

$$\sum_{i''} \omega_{i''} \Delta x_{i''} < \Omega \sum_{i''} \Delta x_{i''} < \Omega \cdot 4\varepsilon.$$

Thus, finally, for $\Delta x_i < \delta$ we have

$$\sum_{i''} \omega_{i''} \Delta x_{i''} < \varepsilon[(b-a) + 4\Omega].$$

This proves our statement, since the term inside the square brackets is a constant number and ε is arbitrarily small.

Finally we give one more simple class of integrable functions; it is not identical with the previous class.

III. *A bounded monotonic function $f(x)$ is always integrable.*

Proof. Suppose that $f(x)$ is a monotonic increasing function. Then its oscillation in the interval $[x_i, x_{i+1}]$ is

$$\omega_i = f(x_{i+1}) - f(x_i).$$

For any $\varepsilon > 0$, set

$$\delta = \frac{\varepsilon}{f(b) - f(a)}.$$

Provided $\Delta x_i < \delta$, we have

$$\sum \omega_i \Delta x_i < \delta \sum [f(x_{i+1}) - f(x_i)] = \delta [f(b) - f(a)] = \varepsilon,$$

which implies the integrability of the function.

Remark. Observe that, *the alteration of the values of an integrable function at a finite (say k) number of points does not affect either the existence or the magnitude of the integral.*

Since this alteration influences not more than k terms of the sum $\sum \omega_i \Delta x_i$, then as $\lambda \to 0$ the sum tends to zero as before. With regard to the magnitude of the integral itself, for both functions—the original and the altered one—the points ξ_i in the integral sum can be chosen so that they do not coincide with the points at which the values of the functions are different.

§ 2. Properties of definite integrals

180. Integrals over an oriented interval. So far, when speaking of a "definite integral in the interval from a to b" we have always understood that $a < b$. We now remove this restriction.

For this purpose we first establish the concept of a *directed* or *oriented* interval. *By an oriented interval $[a, b]$ (where either $a < b$ or $b < a$) we understand the set of values of x which satisfy the inequalities*

$$a \leqslant x \leqslant b \quad \text{or} \quad a \geqslant x \geqslant b$$

respectively, and which are located or ordered from a to b, i.e. in an increasing order if $a<b$ and decreasing if $a>b$. Thus we regard the intervals $[a, b]$ and $[b, a]$ as distinct; their contents are the same but the directions are different.

We may say that the definition of the integral given in Sec. 176 refers to the oriented interval $[a, b]$ only when $a<b$.

Now consider the integral over the oriented interval $[a, b]$, assuming that $a>b$. We may now repeat the usual procedure of subdividing the interval, introducing the points of subdivision in the direction from a to b:

$$a=x_0>x_1>x_2>\ldots>x_i>x_{i+1}>\ldots>x_n=b.$$

Selecting in each subinterval $[x_i, x_{i+1}]$ a point ξ_i in such a way that $x_i\geqslant\xi_i\geqslant x_{i+1}$ we form the integral sum

$$\sigma=\sum_{i=0}^{n-1}f(\xi_i)\Delta x_i,$$

where now all $\Delta x_i=x_{i+1}-x_i<0$. Finally, the limit of this sum for $\lambda=\max\Delta x_i\to 0$ leads to the concept of the integral

$$\int_a^b f(x)dx=\lim_{\lambda\to 0}\sigma.$$

If we take the same points of division and the same points ξ for the intervals $[a, b]$ and $[b, a]$ (where $a\gtrless b$) the corresponding integral sums differ only in sign. Hence, passing to the limits we arrive at the following proposition:

(1) If $f(x)$ is integrable over the interval $[a, b]$ it is also integrable over the interval $[b, a]$, and

$$\int_a^b f(x)dx=-\int_b^a f(x)dx.$$

Incidentally, we could just take this relation as the definition of the integral $\int_a^b$ when $a>b$, assuming that the integral $\int_b^a$ exists.

Observe that according to this definition

$$\int_a^a f(x)dx=0.$$

181. Properties expressed by equalities. We now present some further properties of definite integrals†.

(2) Suppose that the function $f(x)$ is integrable over the greatest of the intervals $[a, b]$, $[a, c]$ and $[c, b]$‡. Then it is integrable in the remaining two intervals and

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,$$

independently of the relative positions of the points a, b, c.

Proof. Suppose first that $a < c < b$ and the function is integrable over the interval $[a, b]$.

Consider a subdivision of the interval $[a, b]$, the point c being taken as one of the points of subdivision. Then, first of all

$$\sum_a^b \omega \Delta x = \sum_a^c \omega \Delta x + \sum_c^b \omega \Delta x\text{§}.$$

Since all the terms on the right are positive and tend to zero, and since all the terms on the left are positive, the latter must also tend to zero; hence the integrability of the function $f(x)$ over the intervals $[a, c]$ and $[c, b]$ is established. Now, evidently,

$$\sum_a^b f(\xi)\Delta x = \sum_a^c f(\xi)\Delta x + \sum_c^b f(\xi)\Delta x.$$

Passing to the limit as $\lambda \to 0$ we arrive at the required relation.

Other relative positions of the points a, b, c can be reduced to the above case. Suppose, for instance, that $b < a < c$ and the function $f(x)$ is integrable in the interval $[c, b]$ or, equivalently, by virtue of *(1)*, over the interval $[b, c]$. By the foregoing proof we have

$$\int_b^c f(x)dx = \int_b^a f(x)dx + \int_a^c f(x)dx,$$

---

† Henceforth if we speak of integral $\int_a^b$ we admit (unless otherwise stated) both the cases $a < b$ and $a > b$.

‡ Alternatively we could assume that $f(x)$ is integrable in each of the two smaller intervals; then it would also be integrable in the greatest one.

§ The meaning of this notation is obvious.

hence taking the first and second integrals to the other side of the relation and changing the limits (using property *(1)*) we arrive at the required relation.

(3) If $f(x)$ is integrable over the interval $[a, b]$, then $k \cdot f(x)$ (where k is a constant) is also integrable over this interval. Moreover,

$$\int_a^b k \cdot f(x)\,dx = k \cdot \int_a^b f(x)\,dx.$$

(4) If $f(x)$ and $g(x)$ are both integrable over the interval $[a, b]$, then $f(x) \pm g(x)$ is also integrable over this interval, and

$$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$$

The proofs of these two results are similar; thus we shall only prove the latter.

Arbitrarily subdivide the interval $[a, b]$ and form the integral sums for all three integrals. All the points ξ_i of the subintervals are selected arbitrarily but they are the same for the three sums; then we have

$$\sum [f(\xi_i) \pm g(\xi_i)]\,\Delta x_i = \sum f(\xi_i)\,\Delta x_i \pm \sum g(\xi_i)\,\Delta x_i.$$

Now let $\lambda \to 0$; since both sums on the right possess limits, the sum on the left must also possess a limit, which proves the integrability of the functions $f(x) \pm g(x)$. Letting $\lambda \to 0$ in the above relation, we arrive at the required result.

182. Properties expressed by inequalities. So far we have investigated properties of integrals expressed by equalities; we now consider properties expressed by inequalities.

(5) If the function $f(x)$ is integrable and non-negative over the interval $[a, b]$, and $a < b$, then

$$\int_a^b f(x)\,dx \geqslant 0.$$

The *proof* is obvious.

A simple corollary of *(5)* and *(4)* is

(6) If both the functions $f(x)$ and $g(x)$ are integrable over the interval $[a, b]$ and $f(x) \leqslant g(x)$ then also

$$\int_a^b f(x)dx \leqslant \int_a^b g(x)dx$$

assuming that $a < b$.

We only have to apply the preceding property to the difference $g(x) - f(x)$.

(7) Suppose that the function $f(x)$ is integrable over the interval $[a, b]$ and $a < b$; then the function $|f(x)|$ also is integrable in this interval and we have the inequality

$$\left| \int_a^b f(x)dx \right| \leqslant \int_a^b |f(x)|dx.$$

We first establish the existence of the integral of $|f(x)|$. If we take two points x' and x'' in the interval $[x_i, x_{i+1}]$, then [Sec. 8]

$$\big||f(x'')| - |f(x')|\big| \leqslant |f(x'') - f(x')|.$$

Therefore denoting by ω_i^* the oscillation of the function $|f(x)|$ in the interval $[x_i, x_{i+1}]$, by the definition of oscillation [Sec. 73] we have $\omega_i^* \leqslant \omega_i$ and therefore

$$0 \leqslant \sum \omega_i^* \Delta x_i \leqslant \sum \omega_i \Delta x_i \dagger,$$

and since the sum on the right tends to zero, the sum on the left must also tend to zero.

The inequality can easily be derived directly by taking the integral sum

$$\left| \sum f(\xi_i) \Delta x_i \right| \leqslant \sum |f(\xi_i)| \Delta x_i \dagger$$

and then passing to the limit.

(8) If $f(x)$ is integrable over $[a, b]$ where $a < b$, and if the inequality

$$m \leqslant f(x) \leqslant M$$

† Since $a < b$, all $\Delta x_i > 0$.

holds over the whole interval, then

$$m(b-a) \leqslant \int_a^b f(x)\,dx \leqslant M(b-a).$$

We can apply property *(6)* to the functions m, $f(x)$ and M, but it is simpler to make use of the obvious inequalities

$$m \sum \Delta x_i \leqslant \sum f(\xi_i)\Delta x_i \leqslant M \sum \Delta x_i$$

and then pass to the limit.

The above relationships can be put in a more convenient form by eliminating the restriction $a < b$.

(9) MEAN VALUE THEOREM. *Suppose that $f(x)$ is integrable over $[a, b]$ $(a \gtrless b)$ and $m \leqslant f(x) \leqslant M$ over the whole interval; then*

$$\int_a^b f(x)\,dx = \mu(b-a),$$

where $m \leqslant \mu \leqslant M$.

Proof. If $a < b$, then by the property *(8)* we have

$$m \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant M.$$

Setting

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \mu,$$

we arrive at the required relation.

For the case $a > b$ the same reasoning can be applied to $\int_b^a$; then, changing the limits, we obtain the previous formula.

The above relation takes a particularly simple form when the function $f(x)$ is continuous. In fact, if we assume that m and M are the smallest and the greatest values of the function, respectively (the existence of which follows from the Weierstrass theorem [Sec. 73]), then by the Bolzano–Cauchy theorem [Sec. 70], the function will take intermediate value μ at a point c of the interval $[a, b]$. Thus

$$\int_a^b f(x)\,dx = (b-a)f(c),$$

c being within $[a, b]$.

The geometric interpretation of this formula is clear. Suppose that $f(x) \geqslant 0$. Consider a curvilinear figure $ABCD$ (Fig. 67) under the curve $y = f(x)$. The area of this figure (expressed by a definite integral) is equal to the area of the rectangle with the same basis and a mean ordinate LM as the height.

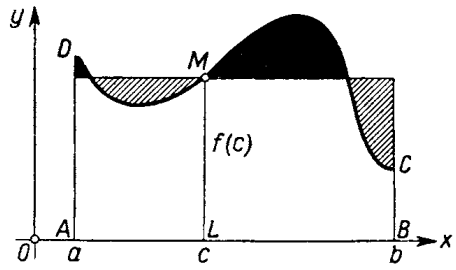

FIG. 67.

(10) GENERALIZED MEAN VALUE THEOREM. *Suppose that* $\{1\}$ $g(x)$ *and the product* $f(x)\,g(x)$ *are integrable over the interval* $[a, b]$; $\{2\}$ $m \leqslant f(x) \leqslant M$; $\{3\}$ $g(x)$ *is of constant sign over the whole interval:* $g(x) \geqslant 0$ $(g(x) \leqslant 0)$. *Then*

$$\int_a^b f(x)\,g(x)\,dx = \mu \int_a^b g(x)\,dx,$$

where $m \leqslant \mu \leqslant M$.

Proof. Suppose, first, that $g(x) \geqslant 0$ and $a < b$; then we have

$$m g(x) \leqslant f(x)\,g(x) \leqslant M g(x).$$

By the properties *(6)* and *(3)* this inequality implies that

$$m \int_a^b g(x)\,dx \leqslant \int_a^b f(x) g(x)\,dx \leqslant M \int_a^b g(x)\,dx.$$

By the assumption concerning the function $g(x)$, from *(5)* we have

$$\int_a^b g(x)\,dx \geqslant 0.$$

If this integral vanishes it follows from the preceding inequalities that at the same time

$$\int_a^b f(x) g(x)\,dx = 0$$

and the theorem is obvious. If the integral is greater than zero, then dividing through by it in the above double inequality and letting

$$\frac{\int_a^b f(x)\,g(x)dx}{\int_a^b g(x)dx} = \mu,$$

we arrive at the required result.

In fact, the restrictions $a<b$ and $g(x)\geqslant 0$ are unnecessary: changing the limits or the sign of $g(x)$ does not alter the relation.

If $f(x)$ is continuous the formula can be written in the form

$$\int_a^b f(x)\,g(x)dx = f(c)\int_a^b g(x)dx,$$

c being in $[a, b]$.

Remark. The variable of integration has constantly been denoted by the letter x; it is evident, however, that the integral would be in no way affected if x were replaced by another letter, provided the limits a and b and the integrand f were unaltered. The symbol $\int_a^b f(x)dx$ denotes exactly the same number as $\int_a^b f(t)dt$ or $\int_a^b f(z)dz$, etc. This obvious remark will often be used below.

183. Definite integral as a function of the upper limit. If the function $f(x)$ is integrable over the interval $[a, b]$ $(a\gtrless b)$. then [Sec. 181, *(2)*] it is also integrable over the interval $[a, x]$, x being an arbitrary value contained in $[a, b]$. Replacing the limit b of the integral by the variable x we obtain the expression

$$\Phi(x) = \int_a^x f(t)dt^{\dagger}, \tag{1}$$

which is evidently a function of x. This function has the following properties:

(11) If the function $f(x)$ is integrable over $[a, b]$, then $\Phi(x)$ is a continuous function of x over the same interval.

---

† The variable of integration has been denoted here by the letter t, to prevent it from having been mistaken for the upper limit x.

Proof. Let x have an arbitrary increment $\Delta x = h$ such that $x + h$ is inside the interval $[a, b]$; the function (1) now has the value

$$\Phi(x+h) = \int_a^{x+h} f(t)\,dt = \int_a^x + \int_x^{x+h}$$

(see *(2)*); hence

$$\Phi(x+h) - \Phi(x) = \int_x^{x+h} f(t)\,dt.$$

Let us now apply the mean value theorem *(9)* to this integral:

$$\Phi(x+h) - \Phi(x) = \mu h; \tag{2}$$

here μ lies between the exact bounds m' and M' of the function $f(x)$ in the interval $[x, x+h]$ and, consequently, between the (constant) bounds m and M over the basic interval $[a, b]$†.

If, now, h tends to zero it is evident that

$$\Phi(x+h) - \Phi(x) \to 0 \text{ or } \Phi(x+h) \to \Phi(x),$$

which proves the continuity of the function $\Phi(x)$.

(12) If we assume that the function $f(t)$ is continuous at the point $t = x$, then the function $\Phi(x)$ has a derivative at this point and

$$\Phi'(x) = f(x)‡.$$

Proof. In fact, from (2) we have

$$\frac{\Phi(x+h) - \Phi(x)}{h} = \mu, \quad \text{where} \quad m' \leqslant \mu \leqslant M'.$$

But by the continuity of the function $f(t)$ at $t = x$, given an arbitrary $\varepsilon > 0$ a number $\delta > 0$ can be found such that for $|h| < \delta$ we have

$$f(x) - \varepsilon < f(t) < f(x) + \varepsilon$$

for all values of t in the interval $[x, x+h]$. In this case the following inequalities hold [Sec. 6]:

$$f(x) - \varepsilon \leqslant m' \leqslant M' \leqslant f(x) + \varepsilon,$$

† It is to be borne in mind that the integrated function is bounded [Sec. 176].

‡ This important proposition was first strictly proved by Cauchy (in 1823) for a function continuous in the whole interval. If we remember a geometrical interpretation of the definite integral as an area [Sec. 175], then theorem *(12)* will be identified with the so-called Newton–Leibniz theorem [Sec. 156].

and hence

$$f(x)-\varepsilon \leqslant \mu \leqslant f(x)+\varepsilon \quad \text{or} \quad |\mu - f(x)| \leqslant \varepsilon.$$

It is now clear that

$$\Phi'(x) = \lim_{h\to 0} \frac{\Phi(x+h)-\Phi(x)}{h} = \lim_{h\to 0} \mu = f(x).$$

This completes the proof.

We have obtained here a result of particular theoretical and applicational value. If we assume that the function $f(x)$ is continuous over the whole interval $[a, b]$, then it is integrable [Sec. 179, I] and *(12)* is applicable at an arbitrary point x of this interval; the derivative of the integral (1) with respect to the upper limit x is everywhere equal to the value $f(x)$ of the integrand at the considered limit.

In other words the primitive function always exists for a function $f(x)$ continuous over an interval $[a, b]$; an example is the integral (1) with a variable upper limit.

Thus we have finally established the proposition mentioned in Sec. 156.

In particular we can now write the F and E functions of Legendre [Sec. 174] in the form of definite integrals:

$$F(k,\varphi) = \int_0^{\varphi} \frac{d\theta}{\sqrt{(1-k^2\sin^2\theta)}}, \qquad E(k,\varphi) = \int_0^{\varphi} \sqrt{(1-k^2\sin^2\theta)}\, d\theta.$$

By the statement just proved these are primitive functions for the functions

$$\frac{1}{\sqrt{(1-k^2\sin^2\varphi)}}, \qquad \sqrt{(1-k^2\sin^2\varphi)},$$

respectively; they vanish for $\varphi = 0$.

Remark. The statements proved in this section can easily be extended to the case of an integral with a variable lower limit, since by *(1)*,

$$\int_x^b f(t)\,dt = -\int_b^x f(t)\,dt.$$

It is evident that the derivative of this integral with respect to x is equal to $-f(x)$ if x is a point of continuity.

§ 3. Evaluation and transformation of definite integrals

184. Evaluation using integral sums. We now give some examples of the evaluation of definite integrals by the direct consideration of the limits of the integral sums. Knowing beforehand that the integral of a continuous function exists, to evaluate it we can use any subdivision of the interval and the points ξ, our choice being governed by convenience only.

(1) $\int_a^b \sin x\,dx$. Dividing the interval $[a,b]$ into n equal parts we set $h = (b-a)/n$; the function $\sin x$ is evaluated at the right-hand ends of the subintervals if $a<b$ and at the left-hand ends if $a>b$. Then

$$\sigma_n = h\sum_{i=1}^{n}\sin(a+ih).$$

Let us sum the finite series on the right. Multiplying and dividing it by $2\sin(h/2)$ and expressing the product terms as the differences of two cosines we easily find that

$$\begin{aligned}\sum_{i=1}^{n}\sin(a+ih) &= \frac{1}{2\sin\frac{1}{2}h}\sum_{i=1}^{n}2\sin(a+ih)\sin\tfrac{1}{2}h\\ &= \frac{1}{2\sin\frac{1}{2}h}\sum_{i=1}^{n}[\cos(a+\overline{i-\tfrac{1}{2}}h)-\cos(a+\overline{i+\tfrac{1}{2}}h)]\\ &= \frac{\cos(a+\frac{1}{2}h)-\cos(a+\overline{n+\frac{1}{2}}h)}{2\sin\frac{1}{2}h}. \qquad (1)\end{aligned}$$

Hence

$$\sigma_n = \frac{\frac{1}{2}h}{\sin\frac{1}{2}h}[\cos(a+\tfrac{1}{2}h)-\cos(b+\tfrac{1}{2}h)].$$

As $h\to 0$, $n\to\infty$, so

$$\int_a^b \sin x\,dx = \lim_{h\to 0}\frac{\frac{1}{2}h}{\sin\frac{1}{2}h}[\cos(a+\tfrac{1}{2}h)-\cos(b+\tfrac{1}{2}h)] = \cos a-\cos b.$$

(2) $\int_a^b x^\mu\,dx$ $(b>a>0;$ μ is an arbitrary real number).

This time we divide the interval $[a,b]$ into unequal parts; between a and b we introduce $n-1$ geometric means. In other words, setting

$$q = q_n = \sqrt[n]{\left(\frac{b}{a}\right)},$$

we consider the sequence of numbers

$$a,\ aq,\ \ldots,\ aq^i,\ \ldots,\ aq^n = b.$$

Observe that as $n \to \infty$ the ratio $q = q_n \to 1$, while the differences $aq^{i+1} - aq$ are all less than $b(q-1) \to 0$.

Calculating the function at the left-hand ends of the subintervals we have

$$\sigma_n = \sum_{i=0}^{n-1} (aq^i)^\mu\,(aq^{i+1} - aq^i) = a^{\mu+1}(q-1)\sum_{i=0}^{n-1} (q^{\mu+1})^i.$$

Assume first that $\mu \neq -1$; then

$$\sigma_n = a^{\mu+1}(q-1)\,\frac{\left(\frac{b}{a}\right)^{\mu+1} - 1}{q^{\mu+1} - 1} = (b^{\mu+1} - a^{\mu+1})\,\frac{q-1}{q^{\mu+1}-1};$$

making use of the known limit [Sec. 65, *(3)*] we have

$$\int_a^b x^\mu\,dx = \lim_{n\to\infty} \sigma_n = (b^{\mu+1} - a^{\mu+1}) \lim_{q\to 1} \frac{q-1}{q^{\mu+1}-1} = \frac{b^{\mu+1} - a^{\mu+1}}{\mu+1}.$$

In the case $\mu = -1$ we obtain

$$\sigma_n = n(q_n - 1) = n\left[\sqrt[n]{\left(\frac{b}{a}\right)} - 1\right],$$

and on the basis of another familiar result [Sec. 65, *(2)*]

$$\int_a^b \frac{dx}{x} = \lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} n\left[\sqrt[n]{\left(\frac{b}{a}\right)} - 1\right] = \log b - \log a.$$

185. The fundamental formula of integral calculus. We know from Sec. 183 that for a function $f(x)$ continuous over the interval $[a, b]$ the integral

$$\Phi(x) = \int_a^x f(t)\,dt$$

is a primitive function. If $F(x)$ is an arbitrary primitive for $f(x)$ (for instance, the one found by the methods of §§1–4 of the preceding chapter), then [Sec. 155]

$$\Phi(x) = F(x) + C.$$

The constant C can easily be determined by setting $x = a$ or $\Phi(a) = 0$; thus we have

$$0 = \Phi(a) = F(a) + C, \quad \text{whence} \quad C = -F(a).$$

Finally,

$$\Phi(x) = F(x) - F(a).$$

In particular, for $x = b$, we obtain

$$\Phi(b) = \int_a^b f(x)\,dx = F(b) - F(a). \tag{A}$$

This is the fundamental formula of the integral calculus†.

Thus, *the value of the definite integral is equal to the difference of the two values at $x = b$ and $x = a$ of an arbitrary primitive function.*

Formula (A) gives an effective and simple method for the evaluation of a definite integral of a continuous function $f(x)$. In fact, for a number of simple classes of these functions we are able to express the primitive functions in a finite form in terms of elementary functions. In these cases the definite integral is evaluated directly by means of the fundamental formula. The difference on the right is usually denoted by the symbol $F(x)\Big|_a^b$ ("double substitution from a to b") and the formula is written in the form

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b. \tag{A*}$$

For instance, we have

$$(1)\quad \int_a^b \sin x\,dx = -\cos x\Big|_a^b = \cos a - \cos b,$$

$$(2)\quad \int_a^b x^\mu\,dx = \frac{x^{\mu+1}}{\mu+1}\Big|_a^b = \frac{b^{\mu+1} - a^{\mu+1}}{\mu+1} \qquad (\mu \neq -1),$$

$$(3)\quad \int_a^b \frac{dx}{x} = \log x\Big|_a^b = \log b - \log a \qquad (a > 0,\ b > 0).$$

These results were derived in the preceding section with more difficulty.

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† The reasoning is entirely analogous to that employed in Sec. 156 for the computation of the function $P(x)$ and the area P. The formula (A) could easily be deduced from the results of Secs. 156 and 175.

186. The formula for the change of variable in a definite integral. The fundamental formula (A) enables us to establish the following rule for the change of variable in an integrand.

It is required to evaluate the integral $\int_a^b f(x)dx$ where $f(x)$ is a function continuous over the interval $[a, b]$. Set $x = \varphi(t)$, where the function $\varphi(t)$ is subject to the following restrictions:

(1) $\varphi(t)$ is defined and continuous over an interval $[\alpha, \beta]$ and its values remain within the bounds of the interval $[a, b]$†, when t ranges in $[\alpha, \beta]$.

(2) $\varphi(\alpha) = a, \varphi(\beta) = b$.

(3) The continuous derivative $\varphi'(t)$ exists over $[\alpha, \beta]$.

Then the following formula holds:

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt. \tag{2}$$

In view of the assumption of the continuity of the integrands, not only do these definite integrals exist but the corresponding indefinite ones also exist, and in both cases we may use the basic formula. But if $F(x)$ is a primitive function for the differential $f(x)dx$, then the function $\Phi(t) = F(\varphi(t))$ is a primitive function for the differential $f(\varphi(t))\varphi'(t)dt$ [Sec. 160]. Hence we simultaneously have

$$\int_a^b f(x)dx = F(b) - F(a)$$

and

$$\int_\alpha^\beta f(\varphi(t))\varphi'(t)dt = \Phi(\beta) - \Phi(\alpha) = F(\varphi(\beta)) - F(\varphi(\alpha)) = F(b) - F(a),$$

which implies the required relation.

Remark. We note an important feature of formula (2). In evaluating an indefinite integral by means of the change of variable method, after obtaining the required function expressed in terms of the new

† It may happen that the function $f(x)$ is defined and continuous over a wider interval $[A, B]$ than $[a, b]$, and then it is sufficient to require that the values of $\varphi(t)$ remain within the bounds of the interval $[A, B]$.

variable we had to return to the previous variable x; however, this is no longer necessary. If the second definite integral (2) can be evaluated (it is a number), then the first one is also known.

Examples. (1) We evaluate the integral $\int_0^a \sqrt{(a^2-x^2)}\,dx$ by means of the substitution $x = a\sin t$; α and β are now equal to 0 and $\pi/2$ respectively. We have

$$\int_0^a \sqrt{(a^2-x^2)}\,dx = a^2\int_0^{\frac{\pi}{2}} \cos^2 t\,dt = \frac{a^2}{2}\left(t+\frac{\sin 2t}{2}\right)\Big|_0^{\frac{\pi}{2}} = \frac{\pi a^2}{4}$$

[see Sec. 160].

(2) Consider the integral

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} = \int_0^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\pi}.$$

The substitution $x = \pi - t$ (t varying from $\pi/2$ to 0) leads to the form

$$-\int_{\frac{\pi}{2}}^{0} \frac{(\pi-t)\sin t}{1+\cos^2 t}\,dt = \int_0^{\frac{\pi}{2}} \frac{(\pi-t)\sin t}{1+\cos^2 t}\,dt,$$

which is equal to

$$\pi\int_0^{\frac{\pi}{2}} \frac{\sin t}{1+\cos^2 t}\,dt - \int_0^{\frac{\pi}{2}} \frac{t\sin t}{1+\cos^2 t}\,dt.$$

Substituting, after transforming we have

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \pi\int_0^{\frac{\pi}{2}} \frac{\sin t}{1+\cos^2 t}\,dt = -\pi\arctan(\cos t)\Big|_0^{\frac{\pi}{2}} = \frac{\pi^2}{4}.$$

187. Integration by parts in a definite integral. We considered in Sec. 162 the formula of integration by parts

$$\int u\,dv = uv - \int v\,du, \tag{3}$$

assuming that the functions u, v of the independent variable x, and their derivatives u', v', are continuous over the considered interval. Now, with the aid of the fundamental formula (A) we

transform formula (3) to an analogous formula containing definite integrals; this reduces the evaluation of one definite integral to the evaluation of another one (generally a simpler one).

Denote the last integral in formula (3) by $\varphi(x)$. Then, by formula (A)

$$\int_a^b u\,dv = [uv - \varphi(x)]\Big|_a^b = uv\Big|_a^b - \varphi(x)\Big|_a^b.$$

Since, again using (A),

$$\int_a^b v\,du = \varphi(x)\Big|_a^b,$$

we finally arrive at the formula

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du. \tag{4}$$

This formula, which is a relation between numbers, is in principle simpler than formula (3) in which functions appear; it is especially useful if the double substitution vanishes.

Example. Evaluate the integrals

$$J_m = \int_0^{\frac{\pi}{2}} \sin^m x\,dx, \qquad J'_m = \int_0^{\frac{\pi}{2}} \cos^m x\,dx$$

for a positive integer m.

Integrating by parts we have

$$J_m = \int_0^{\frac{\pi}{2}} \sin^{m-1} x\,d(-\cos x) = -\sin^{m-1} x\cos x\,\Big|_0^{\frac{\pi}{2}} + (m-1)\int_0^{\frac{\pi}{2}} \sin^{m-2} x\cos^2 x\,dx.$$

The double substitution vanishes. Replacing $\cos^2 x$ by $1-\sin^2 x$ we obtain

$$J_m = (m-1)\,J_{m-2} - (m-1)\,J_m;$$

this gives the *recurrence formula*

$$J_m = \frac{m-1}{m}\,J_{m-2},$$

which successively reduces the integral J_m to J_0 or J_1. Thus, for $m = 2n$ we have

$$J_{2n} = \int_0^{\frac{\pi}{2}} \sin^{2n} x\,dx = \frac{(2n-1)(2n-3)\ldots 3\cdot 1}{2n(2n-2)\ldots 4\cdot 2}\cdot\frac{\pi}{2},$$

while if $m = 2n+1$ we have

$$J_{2n+1} = \int_0^{\frac{\pi}{2}} \sin^{2n+1} x\, dx = \frac{2n(2n-2)\ldots 4\cdot 2}{(2n+1)(2n-1)\ldots 3\cdot 1}.$$

The same results follow for J_m'†.

To abbreviate the notation of the derived expressions we introduce the symbol $m!!$ (where m is an integer); it denotes the product of the positive even (odd) integers not greater than m for m even (odd) (for instance, $6!! = 2\cdot 4\cdot 6$, $7!! = 1\cdot 3\cdot 5\cdot 7$). Then we can write

$$\int_0^{\frac{\pi}{2}} \sin^m x\, dx = \int_0^{\frac{\pi}{2}} \cos^m x\, dx = \begin{cases} \dfrac{(m-1)!!}{m!!}\cdot\dfrac{\pi}{2} & \text{for } m \text{ even} \\ \dfrac{(m-1)!!}{m!!} & \text{for } m \text{ odd.} \end{cases} \tag{5}$$

188. Wallis's formula. It is easy to derive from (5) the celebrated Wallis formula which was announced in 1655 in his *Arithmetic of Infinite Quantities.*

Assuming that $0 < x < \pi/2$ we have the inequalities

$$\sin^{2n+1} x < \sin^{2n} x < \sin^{2n-1} x.$$

Integrating over the interval from 0 to $\pi/2$ we obtain

$$\int_0^{\frac{\pi}{2}} \sin^{2n+1} x\, dx \leqslant \int_0^{\frac{\pi}{2}} \sin^{2n} x\, dx \leqslant \int_0^{\frac{\pi}{2}} \sin^{2n-1} x\, dx.$$

Hence, by (5) we find

$$\frac{2n!!}{(2n+1)!!} < \frac{(2n-1)!!}{2n!!}\,\frac{\pi}{2} < \frac{(2n-2)!!}{(2n-1)!!}$$

or

$$\left[\frac{2n!!}{(2n-1)!!}\right]^2 \frac{1}{2n+1} < \frac{\pi}{2} < \left[\frac{2n!!}{(2n-1)!!}\right]^2 \frac{1}{2n}.$$

Since the difference between the outside quantities

$$\frac{1}{(2n+1)2n}\left[\frac{2n!!}{(2n-1)!!}\right]^2 < \frac{1}{2n}\,\frac{\pi}{2}$$

obviously tends to zero when $n \to \infty$, $\pi/2$ is their common limit. Thus

$$\frac{\pi}{2} = \lim_{n\to\infty}\left[\frac{2n!!}{(2n-1)!!}\right]^2 \frac{1}{2n+1}$$

---

† Observe that J_m' is transformed into J_m by means of the substitution $x = (\pi/2) - t$.

or

$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{2\cdot 2\cdot 4\cdot 4\cdot \ldots \cdot 2n\cdot 2n}{1\cdot 3\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)\cdot (2n+1)}.$$

This is the *Wallis formula*†. It is of historical interest as the first representation of number π in the form of a limit of an easily computable rational variable. In theoretical investigations it is used even now, but for an approximate computation of the number π new methods exist which make it possible to achieve the goal much more rapidly.

§ 4. Approximate evaluation of integrals

189. The trapezium formula. Suppose that it is required to evaluate the definite integral $\int_a^b f(x)dx$ where $f(x)$ is a continuous function defined over the interval $[a, b]$. In § 3 we evaluated a similar integral by means of formula (A) with the aid of the primitive function. But the latter can be expressed in a finite form for only a narrow class of functions; otherwise we have to employ various methods of approximate calculations. These methods yield an approximate

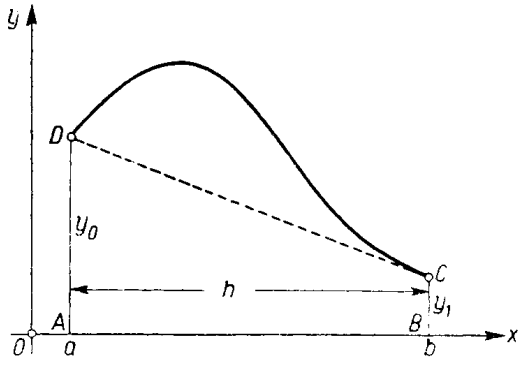

FIG. 68.

expression for the integral in terms of the integrand evaluated for a number of values of the independent variable. In the simplest cases the derivation of such an expression is facilitated by a geometric reasoning, since the definite integral may be interpreted as the area of "the curvilinear trapezium $ABCD$" (Fig. 68) bounded

† Originally it was given for $4/\pi$.

by the curve $y = f(x)$ [Sec. 175] and our problem is reduced to the approximate calculation of this area.

As a first approximation it is natural to replace the curve CD by its chord and the curvilinear trapezium by an ordinary trapezium. To determine the area of the latter it is sufficient to know just the initial and final ordinates

$$f(a) = y_0, \quad f(b) = y_1$$

and the base $b - a = h$. Thus we arrive at the approximate formula

$$\int_a^b f(x)dx \doteq \frac{b-a}{2}[f(a)+f(b)] = \frac{h}{2}(y_0+y_1). \tag{1}$$

Obviously this formula gives only a rough approximation. To derive a more exact one we subdivide the interval $[a, b]$, by means of the points $x_1, x_2, \ldots, x_{n-1}$, into n equal subintervals

$$[a, x_1], [x_1, x_2], \ldots, [x_{n-1}, b] \tag{2}$$

and we construct the corresponding ordinates; the latter divide our figure into n strips. Each of these strips we replace by a trapezium, (Fig. 69) as was done above for the whole figure.

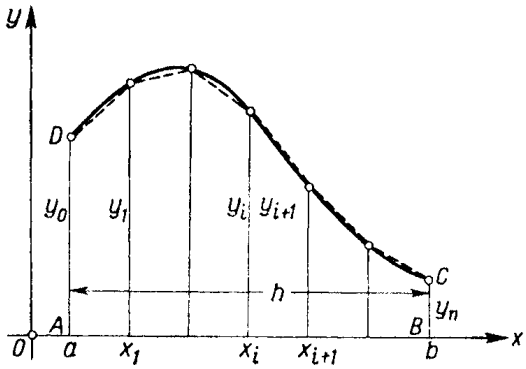

FIG. 69.

Since the heights of all the trapezia are equal to h/n, assuming

$$f(a) = y_0, \quad f(x_1) = y_1, \quad \ldots,$$
$$f(x_{n-1}) = y_{n-1}, \quad f(b) = y_n,$$

we have for the successive areas of the trapezia the values

$$\frac{h}{2n}(y_0+y_1), \quad \frac{h}{2n}(y_1+y_2), \ \ldots, \quad \frac{h}{2n}(y_{n-1}+y_n).$$

Adding we arrive at the approximate formula,

$$\int_a^b f(x)\,dx \doteq \frac{h}{n}\left[\frac{y_0+y_n}{2}+y_1+\ \ldots\ +y_n\right]. \tag{3}$$

This is the so-called *trapezium formula.*

It can be proved that as n increases to infinity the error of the formula decreases to zero. Thus, for a sufficiently large n the formula gives the required value of the integral to an arbitrary degree of accuracy.

As an example consider the familiar integral

$$\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4} = 0.785398\ldots$$

and apply the above approximate formula to it, taking $n = 10$ and calculating to four decimal places.

In accordance with the trapezium formula we have

$$x_0 = 0.0 \qquad y_0 = 1.0000$$
$$x_{10} = 1.0 \qquad y_{10} = 0.5000$$
$$\text{Sum: } 1.5000$$

$$\frac{1}{10}\left(\frac{1.5000}{2}+7.0998\right) = 0.78498$$

$$x_1 = 0.1 \qquad y_1 = 0.9901$$
$$x_2 = 0.2 \qquad y_2 = 0.9615$$
$$x_3 = 0.3 \qquad y_3 = 0.9174$$
$$x_4 = 0.4 \qquad y_4 = 0.8621$$
$$x_5 = 0.5 \qquad y_5 = 0.8000$$
$$x_6 = 0.6 \qquad y_6 = 0.7353$$
$$x_7 = 0.7 \qquad y_7 = 0.6711$$
$$x_8 = 0.8 \qquad y_8 = 0.6098$$
$$x_9 = 0.9 \qquad y_9 = 0.5525$$
$$\text{Sum: } 7.0998$$

The approximate result differs from the true value by less than 0.0005.

Obviously the reader will realize that the error could be estimated only if we knew beforehand the exact value of the integral. In order that our formula be applicable for approximate calculations it is necessary to possess a convenient expression for the error, thus enabling us not only to estimate the error for a given n but also to select an n which would ensure a predetermined accuracy. We shall return to this problem in Sec. 191.

190. Parabolic formula. We return to the curvilinear figure $ABCD$ and, dividing its base AB into halves, at the point E we construct the ordinate EF (Fig. 70). The ordinates

$$AD = y_0, \quad EF = y_{1/2}, \quad BC = y_1$$

and the base $AB = h$ are assumed to be known. Instead of using the chords CF and FD we now replace the curve CD by an arc of the parabola (with vertical axis) passing through the three points C, F, D, hoping that the parabola is a better approximation to the curve than the broken line CFD.

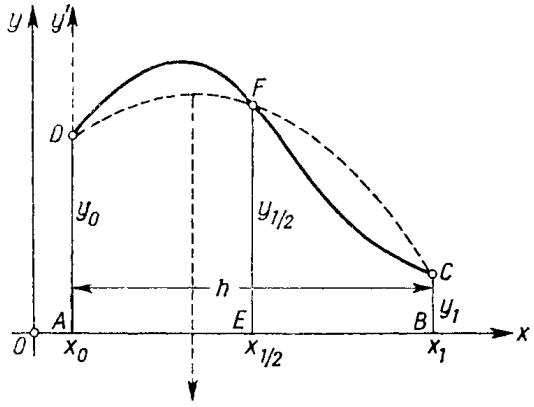

FIG. 70.

Evidently, it is first of all necessary to ensure that through three arbitrary points of the plane

$$(x_0, y_0), \quad (x_{1/2}, y_{1/2}), \quad (x_1, y_1) \quad (x_0 < x_{1/2} < x_1)$$

we can in fact always draw such a parabola and, moreover, that the latter is unique. The equation of a parabola with vertical axis has the form

$$y = ax^2 + bx + c,$$

and the coefficients are uniquely defined by the three conditions

$$ax_0^2 + bx_0 + c = y_0, \quad ax_{1/2}^2 + bx_{1/2} + c = y_{1/2},$$

$$ax_1^2 + bx_1 + c = y_1,$$

since the determinant of the system

$$\begin{vmatrix} x_0^2 & x_0 & 1 \\ x_{1/2}^2 & x_{1/2} & 1 \\ x_1^2 & x_1 & 1 \end{vmatrix}$$

("the Vandermonde determinant") does not vanish†.

We now proceed to calculate the area P of the figure bounded above by the arc of our parabola. We shall show that this area is given by the formula

$$P = \frac{h}{6}(y_0 + 4y_{1/2} + y_1), \tag{4}$$

which result is usually attributed to Simpson‡.

Without affecting the generality we may assume that the y-axis passes through the point A. Then

$$P = \int_0^h (ax^2 + bx + c)\,dx = \frac{h}{6}(2ah^2 + 3bh + 6c).$$

Taking into account that

$$y_0 = c, \quad y_{1/2} = a\frac{h^2}{4} + b\frac{h}{2} + c, \quad y_1 = ah^2 + bh + c,$$

we can directly verify Simpson's formula.

Expression (4), which gives an exact value for the area under the parabola, is only an approximation of the area under the curve $y = f(x)$:

$$\int_a^c f(x)\,dx \doteq \frac{h}{6}(y_0 + 4y_{1/2} + y_1). \tag{5}$$

To increase the accuracy we repeat the process: we divide the interval $[a, b]$ into n equal parts (1) and apply a formula of the type (5) to each of the n strips of which the figure now comprises. Since, besides the extreme values, this formula also contains a mean ordinate, we have to divide each of the n subintervals into halves by means of the points $x_{1/2}, x_{3/2}, \ldots, x_{n-1/2}$ (so that altogether the basic interval

† For $a = 0$ the parabola degenerates into a straight line.

‡ Thomas Simpson (1710–1761)—an English mathematician. Apparently the formula was known before him.

is divided into $2n$ equal parts). Since the bases of all n (not $2n$) strips are equal to h/n, we obtain for the areas of each the approximate expressions

$$\frac{h}{6n}(y_0+4y_{1/2}+y_1),$$

$$\frac{h}{6n}(y_1+4y_{3/2}+y_2),\ \ldots,\ \frac{h}{6n}(y_{n-1}+4y_{n-1/2}+y_n),$$

respectively. Adding, we arrive at a new approximate formula

$$\int_a^b f(x)\,dx \doteq \frac{h}{6n}[(y_0+y_n)+2(y_1+\ldots+y_{n-1}) + 4(y_{1/2}+\ldots+y_{n-1/2}), \quad (6)$$

which is called the *parabolic formula* or *Simpson's formula*; it is more frequently used for the approximate evaluation of integrals than the trapezium formula, since it usually yields a more exact result for the same amount of calculation.

For comparison we again evaluate the integral $\int_0^1 \frac{dx}{1+x^2}$ according to Simpson's formula. We take $2n=4$ and therefore the number of ordinates employed is now less than before. We have (calculating to five decimal places)

$$\begin{array}{lllll} x_0=0 & x_{1/2}=\frac{1}{4} & x_1=\frac{1}{2} & x_{3/2}=\frac{3}{4} & x_2=1 \\ y_0=1 & 4y_{1/2}=3.76471 & 2y_1=1.6 & 4y_{3/2}=2.56 & y_2=0.5 \end{array}$$

$$\tfrac{1}{12}(1+3.76471+1.6+2.56+0.5)=0.78539\ldots.$$

This time all five digits are correct.

Of course, all remarks made at the end of the preceding section regarding formula (5) may be repeated. We now proceed to estimate the errors of the approximate formulae.

191. Remainder term for the approximate formulae. First consider the simplest particular case of the trapezium formula, given by $n=1$, i.e. formula (1). Writing the error as ϱ, we have

$$\int_a^b f(x)\,dx = \frac{b-a}{2}[f(a)+f(b)]+\varrho,$$

and the problem consists in finding for ϱ an expression which enables us to estimate it conveniently.

We assume that the function $f(x)$ has continuous derivatives of the first two orders over the interval $[a, b]$. Then the following elementary transformations

of the integral $\int_a^b f(x)\,dx$, consisting of a triple integration by parts, lead directly to the required expression for ϱ.

We have

$$\int_a^b f(x)\,dx = \int_a^b f(x)\,d(x-a) = f(b)(b-a) - \int_a^b f'(x)(x-a)\,dx,$$

$$\int_a^b f'(x)(x-a)\,dx = \int_a^b f'(x)(x-a)\,d(x-b) = -\int_a^b (x-b)\,d[f'(x)(x-a)]$$

$$= -\int_a^b f''(x)(x-a)(x-b)\,dx - \int_a^b f'(x)(x-b)\,dx,$$

$$\int_a^b f'(x)(x-b)\,dx = \int_a^b (x-b)\,df(x) = f(a)(b-a) - \int_b^b f(x)\,dx.$$

Hence we obtain

$$\int_a^b f(x)dx = (b-a)\,[f(a)+f(b)] - \int_a^b f(x)dx + \int_a^b f''(x)(x-a)(x-b)dx,$$

and therefore

$$\int_a^b f(x)dx = \frac{b-a}{2}\,[f(a)+f(b)] + \frac{1}{2}\int_a^b f''(x)(x-a)(x-b)dx,$$

and

$$\varrho = \frac{1}{2}\int_a^b f''(x)(x-a)(x-b)dx.$$

Since the function $f''(x)$ is continuous and the factor $(x-a)(x-b)$ does not change its sign in the interval $[a, b]$, by the general mean value theorem [Sec. 182, *(10)*] we have

$$\varrho = \frac{1}{2}f''(\bar{\xi})\int_a^b (x-a)(x-b)dx = -\frac{(b-a)^3}{12}f''(\bar{\xi}),$$

where $a \leqslant \bar{\xi} \leqslant b$†.

If the interval $[a, b]$ is divided into $n > 1$ equal parts, then for every subinterval $[x_i, x_{i+1}]$, by the result proved above, we have the *exact* formula

$$\int_{x_i}^{x_{i+1}} f(x)dx = \frac{b-a}{n}\,\frac{y_i+y_{i+1}}{2} - \frac{(b-a)^3}{12n^3}f''(\xi_i) \qquad (x_i \leqslant \xi_i \leqslant x_{i+1}).$$

† This simple derivation of the expression for the additional term of formula (1) was given by the student G. Tseytin.

Adding these relations term by term (for $i = 0, 1, \ldots, n-1$) we obtain

$$\int_a^b f(x)dx = \frac{h}{n}\left(\frac{y_0+y_n}{2} + y_1 + \ldots + y_n\right) + R_n,$$

$$(h = b-a)$$

where the expression

$$R_n = -\frac{h^3}{12n^2}\,\frac{f''(\xi_0)+f''(\xi_1)+\ldots+f''(\xi_{n-1})}{n}$$

is just the remainder term of the trapezium formula (3).

Denoting by m and M the smallest and the greatest values of the continuous function $f''(x)$ over the interval $[a, b]$, respectively [Sec. 73], we find that the arithmetic mean

$$\frac{f''(\xi_0)+f''(\xi_1)+\ldots+f''(\xi_{n-1})}{n}$$

also lies between m and M. According to the familiar property of continuous functions [Sec. 70] there exists in $[a, b]$ a point ξ such that the considered expression is equal to $f''(\xi)$. Consequently we have, finally,

$$R_n = -\frac{h^3}{12n^2}f''(\xi) \qquad (a \leqslant \xi \leqslant b). \tag{7}$$

When n increases, this additional term decreases approximately† as $1/n^2$.

Let us, for instance, return to the evaluation of the integral $\int_0^1 \frac{dx}{1+x^2}$ carried out in Sec. 189. For the integrand $f(x) = 1/(1+x^2)$ we have $f''(x) = 2(3x^2-1)/(1+x^2)^3$; this derivative changes its sign in the interval $[0, 1]$ but its absolute value is less than 2. Hence by formula (7) $|R_{10}| < 0.0017$. We have calculated the ordinate to four decimal places, the accuracy being 0.00005; it is readily observed that the error resulting from the approximation of the ordinate can be included in the above estimate. The true error is in fact smaller than this bound.

For the case of the Simpson formula (6) we shall just give the remainder term without describing its derivation. Assuming that the first four continuous derivatives of the function $f(x)$ exist, the remainder term (if the interval is divided into $2n$ equal parts) has the form

$$R_{2n} = -\frac{h^5}{180\cdot(2n)^4}f^{(4)}(\eta) \qquad (a \leqslant \eta \leqslant b). \tag{8}$$

We again consider the integral $\int_0^1 \frac{dx}{1+x^2}$. To avoid the calculation of the fourth derivative appearing in formula (8) we note that the function $f(x) = 1/(1+x^2)$

† We say approximately since ξ may change with n. This should henceforth be borne in mind.

itself is the derivative of $y = \arctan x$, and, consequently, we can make use of the derived formula of Sec. 96, *(5)*. Thus

$$f^{(4)}(x) = y^{(5)} = 24\cos^5 y \sin 5\left(y + \frac{\pi}{2}\right),$$

whence $|f^{(4)}(x)| \leqslant 24$ and by formula (7) $|R_4| < 1/1920 < 0.0006$. We know that the true error is considerably smaller than this bound.

192. Example. In conclusion, in order to give an example of the approximate evaluation of a definite integral the value of which is not known beforehand, we evaluate a *complete elliptic integral*† of the second kind

$$E\left(\frac{1}{\sqrt{2}}\right) = \int_0^{\frac{\pi}{2}} \sqrt{\left(1 - \frac{1}{2}\sin^2 x\right)}\,dx$$

by the Simpson formula, the required accuracy being 0.001.

For the function $f(x) = \sqrt{[1 - \frac{1}{2}\sin^2 x]}$, when x varies from 0 to $\pi/2$, we have $|f^{(4)}(x)| < 12$‡, and hence (see (7))

$$|R_{2n}| < \frac{\left(\frac{\pi}{2}\right)^5}{180 \cdot (2n)^4} 12 < \frac{2}{3}\frac{1}{(2n)^4}, \quad \text{since} \quad \left(\frac{\pi}{2}\right)^5 < 10.$$

Take $2n = 6$ so that $|R_6| < 0.00052$. Then

$$x_0 = 0 \quad (0°) \qquad y_0 = 1.0000$$

$$x_{1/2} = \frac{\pi}{12}\ (15°) \qquad 4y_{1/2} = \sqrt{(12 + \sqrt{12})} = 3.9324$$

$$x_1 = \frac{\pi}{6}\ (30°) \qquad 2y_1 = \sqrt{14}/2 = 1.8708$$

$$x_{3/2} = \frac{\pi}{4}\ (45°) \qquad 4y_{3/2} = \sqrt{12} = 3.4641 \qquad \frac{\pi}{2}\frac{15.4771}{18} = 1.35063\ldots$$

$$x_2 = \frac{\pi}{3}\ (60°) \qquad 2y_2 = \sqrt{10}/2 = 1.5811$$

$$x_{5/2} = \frac{5\pi}{12}\ (75°) \qquad 4y_{5/2} = \sqrt{(12 - \sqrt{12})} = 2.9216$$

$$x_6 = \frac{\pi}{2}\ (90°) \qquad y_3 = \sqrt{2}/2 = 0.7071$$

$$\text{Sum} \quad 15.4771$$

† By a complete integral we mean the Legendre integrals $F(k, \varphi)$ and $E(k, \varphi)$ for $\varphi = \pi/2$; in this case we omit the second argument symbol and we simply write $F(k)$, $E(k)$. Special tables exist for complete integrals.

‡ Evidently $y = f(x) \geqslant 1/\sqrt{2}$; differentiating the identity

$$y^2 = 1 - \tfrac{1}{2}\sin^2 x,$$

we easily successively obtain the estimates from the above of the absolute values of the derivatives y', y'', y''', y''''.

We should add to the deduced result beside the correction R_6 the (non-negative) approximational correction, which does not exceed $0.0003\ \pi/36 < 0.00003$.

Thus

$$1.35011 < E\left(\frac{1}{\sqrt{2}}\right) < 1.35118,$$

and we may state that $E(1/\sqrt{2}) = 1.351 \pm_{0.001}$.

(In fact, all the digits of the derived result are correct.)

This example is interesting in the respect that the corresponding primitive function cannot be expressed in a finite form and therefore it cannot be employed to evaluate the definite integral.

Conversely, if in this or similar cases the primitive functions are represented in the form of a definite integral with a variable upper limit we could compute the values of these integrals corresponding to a sequence of values of the upper limit. Essentially this explains the possibility of constructing, for functions given by integral expressions only, tables similar to those with which the reader is familiar for elementary functions.

CHAPTER 12

GEOMETRIC AND MECHANICAL APPLICATIONS OF INTEGRAL CALCULUS

§ 1. Areas and volumes

193. Definition of the concept of area. Quadrable domains. By a *polygonal domain* or briefly a *polygon* we shall mean an arbitrary finite (not necessarily connected) plane figure bounded by one or several broken lines. The concept of area for such a figure is fully investigated in school courses of geometry; it will constitute the basis of the present considerations.

Let us take an arbitrary plane figure (P) which represents a closed bounded domain. Its boundary or contour (K) will always be represented in the form of a closed curve (or several such curves).

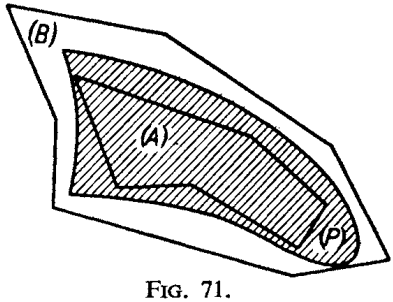

FIG. 71.

We shall examine all the possible polygons (A) which are wholly contained in (P) and all the polygons (B) wholly containing (P) (Fig. 71). If A and B denote their areas, respectively, then $A \leqslant B$. The set of numbers $\{A\}$ bounded above by any B has the least upper bound P_* [Sec. 6], and $P_* \leqslant B$. Similarly, the set of numbers $\{B\}$ bounded below by the number P_* has the greatest lower bound $P^* \geqslant P_*$. These bounds could be called the interior and the exterior areas of the figure (P), respectively.

If the two bounds

$$P_* = \sup \{A\} \quad \text{and} \quad P^* = \inf \{B\}$$

are identical, their common value P is called the *area* of the figure (P). In this case the figure (P) is said to be quadrable (or *squarable*).

(1) A necessary and sufficient condition for the existence of the area of the figure is that for any $\varepsilon > 0$ two polygons (A) and (B) can be found, such that $B - A < \varepsilon$.

In fact, the necessity of the condition follows from the basic properties of the least upper and greatest lower bounds [Sec. 6]; if the area P exists we can find $A > P - \varepsilon/2$ and $B < P + \varepsilon/2$. The sufficiency follows directly from the inequalities

$$A \leqslant P_* \leqslant P^* \leqslant B.$$

The same idea can be expressed in a different form: the curve (K), which is the contour of (P), plays an essential role in the problem of the quadrability of (P).

If the domain is quadrable, then, as we have just seen, corresponding to a given $\varepsilon > 0$ the curve (K) may be included in a polygonal domain $(B - A)$ contained between the contours of the two polygons (A) and (B) (see Fig. 71), and having the area $B - A < \varepsilon$.

Conversely, assume that the contour (K) can be enclosed within a polygonal domain (C) with area $C < \varepsilon$ where ε is an arbitrary positive number. Furthermore, without loss of the generality we may assume that (C) does not cover the whole figure P. Then points of the domain (P) which do not lie within (C) form a polygonal domain (A) contained in (P); if we now join (A) and (C) we obtain a polygonal domain (B) which contains (P). Since the difference $B - A = C < \varepsilon$, by *(1)* this implies the quadrability of the domain (P).

To simplify the terminology we say that a (closed or open) curve (K) has zero area if it can be covered by a polygonal domain with an arbitrarily small area. We can now formulate the condition of quadrability in a new form.

(2) For a figure (P) to be quadrable it is necessary and sufficient that its contour (K) has zero area.

In this connection it becomes important to find wide classes of curves with zero areas.

It can easily be proved that this property is possessed by any continuous curve expressible by an explicit equation of the form

$$y = f(x) \quad \text{or} \quad x = g(y) \tag{1}$$

$$(a \leqslant x \leqslant b) \qquad (c \leqslant y \leqslant d)$$

(where f and g are continuous functions).

Suppose, for instance, that the first equation holds. For a given $\varepsilon > 0$ we can subdivide the interval $[a, b]$ into parts $[x_i, x_{i+1}]$ $(i = 0, 1, \ldots, n-1)$ so that in each of them the oscillation ω_i of the function f is $\varepsilon/(b-a)$ [Sec. 75]. Denoting, as before, the smallest and the greatest values of the function f in the ith interval by m_i and M_i, respectively, the whole curve is covered by a figure of rectangles

$$[x_i, x_{i+1}; m_i, M_i] \qquad (i = 0, 1, \ldots, n-1)$$

(Fig. 72) with the common area

$$\sum_i (M_i - m_i)(x_{i+1} - x_i) = \sum_i \omega_i \Delta x_i < \frac{\varepsilon}{b-a} \sum_i \Delta x_i = \varepsilon,$$

which was to be proved. Consequently curve (1) has zero area. This implies the following:

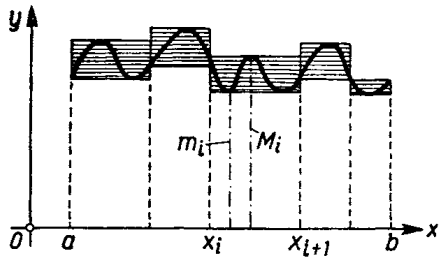

FIG. 72.

(3) If the figure (P) is bounded by a number of continuous curves each being expressed by an explicit equation of either of the types (1), *then the figure is quadrable.*

In fact, since every curve has zero area, then obviously the whole contour also has zero area.

194. The additive property of area. Suppose that the figure (P) is decomposed into two figures (P_1) and (P_2)†; this can be done for instance, by means of a curve connecting two points of the contour and wholly located inside (P) (Figs. 73*a* and *b*). Then the following theorem holds.

(4) Quadrability of any two of the three figures (P), (P_1), (P_2) *implies the quadrability of the third, and*

$$P = P_1 + P_2, \tag{2}$$

i.e. area is *additive*.

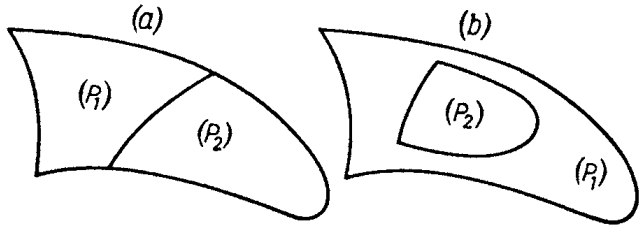

FIG. 73.

The statement concerning the quadrability follows directly from the condition *(2)*. It remains to prove the relation (2). Consider the interior and exterior polygons (A_1), (B_1) and (A_2), (B_2) corresponding to the figures (P_1) and (P_2). The non-overlapping polygons (A_1), (A_2) together constitute a domain (A) with area $A = A_1 + A_2$, which is wholly contained in the domain (P). Now the polygons (B_1) and (B_2), which may overlap, constitute a domain (B) with area $B \leqslant B_1 + B_2$, and which contains the domain (P).

We have

$$A_1 + A_2 \leqslant P \leqslant B \leqslant B_1 + B_2$$

and

$$A_1 + A_2 \leqslant P_1 + P_2 \leqslant B_1 + B_2,$$

and consequently the numbers P and $P_1 + P_2$ lie within the same arbitrarily close bounds $A_1 + A_2$ and $B_1 + B_2$. Therefore these numbers are equal, which completes the proof.

† They can partly have a common boundary but they do not overlap, i.e. they have no common interior points.

Observe that in particular the above results imply that $P_1 < P$ and hence *a part of a figure has an area smaller than the whole figure.*

195. Area as a limit. The condition of quadrability *(1)* formulated in the preceding section can also be stated as follows.

(5) In order that the figure (P) *be quadrable it is necessary and sufficient that there exist two sequences of polygons* $\{(A_n)\}$ *and* $\{(B_n)\}$ *contained in* (P) *and containing* (P)*, respectively, such that their areas have the common limit*

$$\lim A_n = \lim B_n = P. \tag{3}$$

It is evident that this limit is the area of the figure (P).

Sometimes, instead of polygons, it is more convenient to use other figures the quadrability of which has already been established:

(6) If for a figure (P) *we can construct two sequences of quadrable figures* $\{(Q_n)\}$ *and* $\{(R_n)\}$ *contained in* (P) *and containing* (P)*, respectively, the areas of which have a common limit*

$$\lim Q_n = \lim R_n = P,$$

then the figure (P) *is also quadrable and the common limit is its area.*

This follows at once from the preceding statement, if every figure (Q_n) is replaced by a polygon (A_n) contained in it, and the figure (R_n) by a polygon (B_n) containing (R_n), the areas of which being so close that condition (3) is satisfied.

196. An integral expression for area. We now consider the evaluation of plane figures by means of integrals.

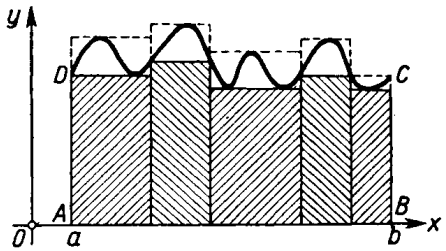

FIG. 74.

We now examine, for the first time in a precise way, the already familiar problem of finding the area of the *curvilinear trapezium ABCD* (Fig. 74). This figure is bounded above by the curve *DC*, which has the equation

$$y = f(x);$$

$f(x)$ being a positive continuous function over the interval $[a, b]$ the figure is bounded below by the segment AB of the x-axis and on the sides by the two ordinates AD and BC (each of the latter may reduce to a point). The actual *existence* of the area P of the figure $ABCD$ follows from *(3)* and we are now concerned with the problem of calculating it.

For this purpose we subdivide the interval $[a, b]$, as before, introducing between a and b the sequence of points

$$a = x_0 < x_1 < x_2 < \ldots < x_i < x_{i+1} < \ldots < x_n = b.$$

Denoting by m_i and M_i, respectively, the smallest and the greatest values of the function $f(x)$ in the ith interval $(i = 0, 1, \ldots, n-1)$ we form the Darboux sums

$$s = \sum_i m_i \Delta x_i, \quad S = \sum_i M_i \Delta x_i.$$

It is evident that they represent the areas of the step figures constructed from the interior and exterior polygons, respectively (see Fig. 74). Hence

$$s < P < S.$$

But when the length of the greatest subinterval Δx_i tends to zero both sums have the limit $\int_a^b f(x)dx$† and consequently this is the required area

$$P = \int_a^b y\,dx = \int_a^b f(x)\,dx. \tag{4}$$

If the curvilinear trapezium $CDFE$ is bounded below and above by curves (see Fig. 75) the equations of which are

$$y_1 = f_1(x) \quad \text{and} \quad y_2 = f_2(x) \quad (a \leqslant x \leqslant b),$$

then regarding it as the difference of two figures $ABFE$ and $ABDC$, we obtain the area of the trapezium (see *(4)*) in the form

$$P = \int_a^b (y_2 - y_1)\,dx = \int_a^b [f_2(x) - f_1(x)]\,dx. \tag{5}$$

† In view of *(5)* this itself proves the quadrability of the curvilinear trapezium $ABCD$; in order to obtain the mentioned sequences of figures we could, for instance, divide the interval into n equal parts, letting n tend to infinity.

Now suppose that a sector AOB (Fig. 76) is bounded by the curve AB and two radii OA and OB (which may both reduce to a point). Then the curve AB is given by a polar equation $r = g(\theta)$ where $g(\theta)$ is a function positive and continuous in the interval $[\alpha, \beta]$.

Introducing between α and β (see Fig. 76) the values of θ

$$\alpha = \theta_0 < \theta_1 < \theta_2 < \ldots < \theta_i < \theta_{i+1} < \ldots < \theta_n = \beta,$$

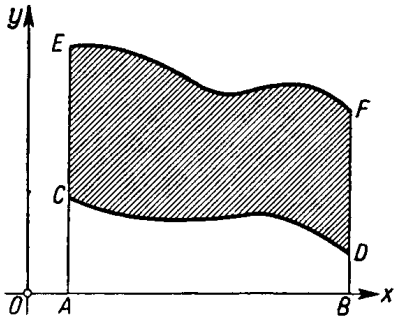

FIG. 75.

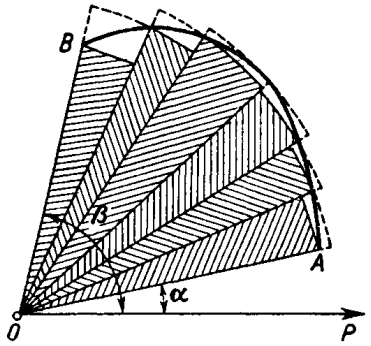

FIG. 76.

we construct the radii corresponding to these angles. Let μ_i and M_i be the smallest and greatest values of the function $g(\theta)$ over $[\theta_i, \theta_{i+1}]$; the sectors corresponding to the radii $\theta = \alpha, \theta = \beta$ are the interior and the exterior sectors, respectively, for the figure (P). Let us construct separately from the interior and exterior sectors two figures the areas of which are

$$\sigma = \frac{1}{2}\sum_i \mu_i^2 \Delta\theta_i \quad \text{and} \quad \Sigma = \frac{1}{2}\sum_i M_i^2 \Delta\theta_i.$$

We easily recognize these sums σ and Σ as being the Darboux sums for the integral $\frac{1}{2}\int_\alpha^\beta [g(\theta)]^2 d\theta$; when the greatest difference $\Delta\theta_i$ tends to zero they both have the above integral as the limit. Consequently, in view of *(6)*† figure (P) is quadrable and

$$P = \frac{1}{2}\int_\alpha^\beta r^2 d\theta = \frac{1}{2}\int_\alpha^\beta [g(\theta)]^2 d\theta. \tag{6}$$

† To obtain the sequences mentioned in *(6)* we could divide the interval into n equal parts.

Examples. *(1)* The ellipse $x^2/a^2 + y^2/b^2 = 1$ and a point $M(x, y)$ on it are given (Fig. 77). It is required to determine the area of the trapezium $BOKM$ and of the sector OMB.

From the equation of the ellipse we have $y = (b/a)\sqrt{(a^2 - x^2)}$ and by formula (4)

$$P_1 = \text{area } BOKM = \int_0^x \frac{b}{a}\sqrt{(a^2 - x^2)}\,dx$$

$$= \frac{ab}{2}\arcsin\frac{x}{a} + \frac{b}{2a}x\sqrt{(a^2 - x^2)} = \frac{ab}{2}\arcsin\frac{x}{a} + \frac{xy}{2}.$$

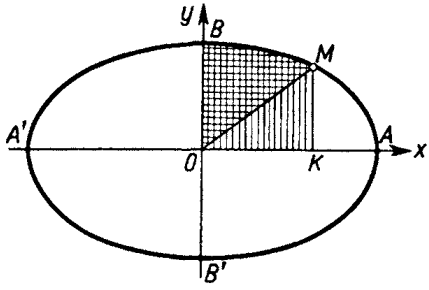

FIG. 77.

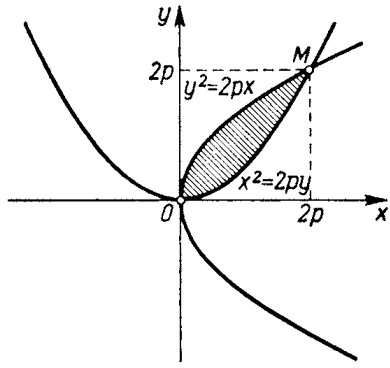

FIG. 78.

Since the last term is the area of ΔOKM, subtracting it we obtain for the area of the sector the expression

$$P_2 = \text{area } OMB = \frac{ab}{2}\arcsin\frac{x}{a}.$$

Putting $x = a$, the area of a quarter of the ellipse is $\pi ab/4$, and the area of the whole ellipse $P = \pi ab$. For a circle $a = b = r$ and so we arrive at the familiar formula $P = \pi r^2$.

(2) We find the area of the figure contained between two parabolas $y^2 = 2px$ and $x^2 = 2py$ (Fig. 78).

Clearly we have to use formula (5), setting

$$y_1 = \frac{x^2}{2p}, \quad y_2 = \sqrt{(2px)}.$$

To find the interval of integration we solve the simultaneous equations and we obtain the abscissa M of the point of intersection (other than the origin) of the parabolas; it is equal to $2p$. We have

$$P = \int_0^{2p}\left(\sqrt{(2px)} - \frac{x^2}{2p}\right)dx = \left(\frac{2}{3}\sqrt{(2p)}x^{\frac{3}{2}} - \frac{x^3}{6p}\right)\Bigg|_0^{2p} = \frac{4}{3}p^2.$$

(3) Formula (4) can also be used in the case when the curve bounding the curvilinear trapezium is given parametrically, for example, by the equations

$$x = \varphi(t), \quad y = \psi(t) \quad (t_0 \leqslant t \leqslant T).$$

Changing the variable in the integral (4) we obtain (assuming that $x = a$ at $t = t_0$ and $x = b$ at $t = T$)

$$P = \int_{t_0}^{T} y x_t' \, dt = \int_{t_0}^{T} \psi(t)\varphi'(t)dt. \tag{7}$$

If, for instance, in finding the area of the ellipse we use the parametric representation

$$x = a\cos t, \quad y = b\sin t,$$

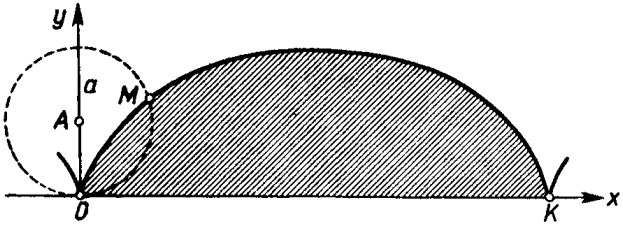

FIG. 79.

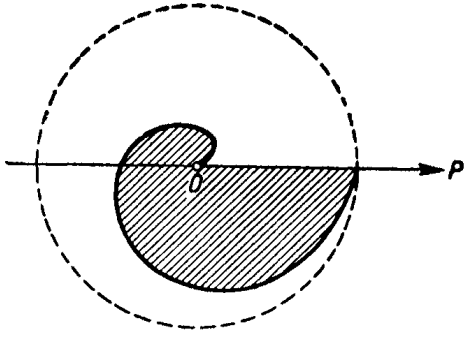

FIG. 80.

and we note that x increases from $-a$ to a when t decreases from π to 0, we find that

$$P = 2\int_{\pi}^{0} b\sin t \cdot (-a\sin t)dt = 2ab\int_{0}^{\pi} \sin^2 t \, dt = \pi ab.$$

Here we found the area of the upper half of the ellipse and then doubled it.

(4) Analogously, we calculate the area of the figure bounded by the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$ (Fig. 79). We have, by (7),

$$P = \int_{0}^{2\pi} a^2(1-\cos t)^2 \, dt = a^2\left(\frac{3}{2}t - 2\sin t + \frac{1}{4}\sin 2t\right)\Bigg|_0^{2\pi} = 3\pi a^2.$$

Thus the required area is equal to three times the area of a circle of radius a.

(5) It is required to find the area of one spire of the Archimedean spiral $r = a\theta$ (Fig. 80).

We have, by (6),

$$P = \frac{1}{2} a^2 \int_0^{2\pi} \theta^2 \, d\theta = \frac{a^2}{6} \theta^2 \Big|_0^{2\pi} = \frac{4}{3} \pi^3 a^2,$$

while the area of the circle of radius $2\pi a$ is $4\pi^3 a^2$. Thus, the area of a spire of the spiral is equal to one-third of the area of the circle (this result was known to Archimedes).

197. Definition of the concept of volume and its properties. As in Sec. 193, where, using the concept of the area of a polygon, we established the concept of the area of an arbitrary plane figure, we now present the definition of the volume of a body on the basis of the volume of a polyhedron.

Thus consider a body (V) of arbitrary form, i.e. a bounded closed domain in three-dimensional space. The boundary (S) of the body is a closed surface (or several such surfaces).

We shall examine polyhedra (X) of volume X wholly contained in the body and polyhedra Y of volume (Y) wholly containing the body. The least upper bound V_* for X and the greater lower bound V^* for Y exist, and moreover $V_* \leqslant V^*$; they could be called the *interior* and *exterior* volumes of the body, respectively.

If both quantities

$$V_* = \sup \{X\} \quad \text{and} \quad V^* = \inf \{Y\}$$

are identical their common value V is called the volume of the body (V). In this case the body (V) is said to be *cubable*.

Here, too, we can easily prove the following theorem.

(1) A necessary and sufficient condition for the existence of the volume of a body is that for any $\varepsilon > 0$ two polyhedra (X) and (Y) can be found, such that $Y - X < \varepsilon$.

This theorem can be given in another form.

(2) In order that a body (V) has a volume it is necessary and sufficient that the bounding surface (S) of the body has zero volume, i.e. that it is possible to include (S) into a polyhedral body with an arbitrary small volume.

First, the surfaces with zero volume are the surfaces expressed by an explicit equation of one of the three types

$$z = f(x, y), \quad y = g(z, x), \quad x = h(y, z),$$

where f, g, h are continuous functions of two arguments in some bounded domains. Suppose that we have an equation of the first type in a domain contained in the rectangle (R). By the theorem of Sec. 137, for any $\varepsilon > 0$ the rectangle can be divided into sufficiently small rectangles (R_i) $(i = 1, 2, \ldots, n)$, such that the oscillation of the function f in the part (P_i) of the domain (P) which is contained in (R_i) is less than ε/R. If m_i and M_i are the smallest and the greatest values of the function f in (P_i) the whole surface can be enclosed within a polyhedron constructed of rectangular parallelepipeds with bases of area R_i and heights $\omega_i = M_i - m_i$. The volume of this polyhedron is

$$\sum_i \omega_i R_i < \frac{\varepsilon}{R} \sum_i R_i = \varepsilon.$$

This completes the proof.

Hence we have

(3) If the body (V) is bounded by several continuous surfaces each of the latter being expressed by an explicit equation (of one of the above three types), then this body always has a volume.

As for area, the volume has the property that it is additive.

(4) If the body (V) is divided into two bodies (V_1) and (V_2), then the existence of the volume for any two of these three bodies implies the existence of the volume for the third one. Then

$$V = V_1 + V_2.$$

It is also easy to state for volumes the propositions analogous to *(5)* and *(6)* of Sec. 195.

(5) In order that the body (V) shall possess a volume it is necessary and sufficient that there exist two sequences of interior and exterior polyhedra $\{(X_n)\}$ and $\{(Y_n)\}$, respectively, the volumes of which have the common limit

$$\lim X_n = \lim Y_n = V.$$

This limit is the volume of the body (V).

It is useful to note a similar proposition concerning, instead of polyhedra, arbitrary bodies which are known to have volumes.

(6) If for the body (V) we can construct two sequences of interior and exterior bodies $\{(T_n)\}$ and $\{(U_n)\}$, respectively, the latter bodies having volumes tending to the common limit

$$\lim T_n = \lim U_n = V,$$

then the body (V) possesses a volume which is equal to the above limit.

198. Integral expression for the volume. We start from an almost obvious remark—a straight cylinder of height H the base of which is a quadrable plane figure (P), has volume equal to the product of the area of the base and the height: $V = PH$.

Take polygons (A_n) and (B_n) contained in (P) and containing (P), respectively, so that their areas A_n and B_n tend to P [Sec. 195, *(5)*]. Constructing on these polygons straight prisms (X_n) and (Y_n) of height H, their volumes

$$X_n = A_n H \quad \text{and} \quad Y_n = B_n H$$

tend to the common limit $V = PH$, which [by Sec. 197, *(5)*] is the volume of the above cylinder.

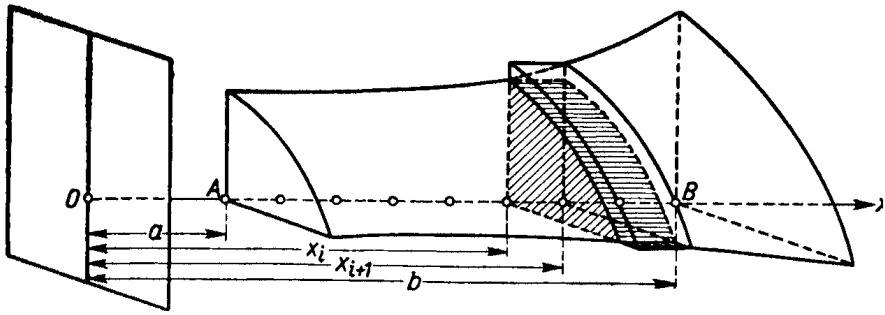

FIG. 81.

Now consider a body (V) contained between the planes $x = a$ and $x = b$ and cut (V) by planes perpendicular to the x-axis (Fig. 81). Assume that all the cross-sections are quadrable and the area of the cross-section corresponding to the abscissa x, denoted by $P(x)$, is a continuous function of x (for $a \leqslant x \leqslant b$).

The projections without deformation of any two of these cross-sections onto a plane perpendicular to the x-axis will lie either inside, or outside, each other (Fig. 82 *b* and *c*).

We examine the case in which the projections of any two distinct cross-sections onto a plane perpendicular to the x-axis lie inside each other.

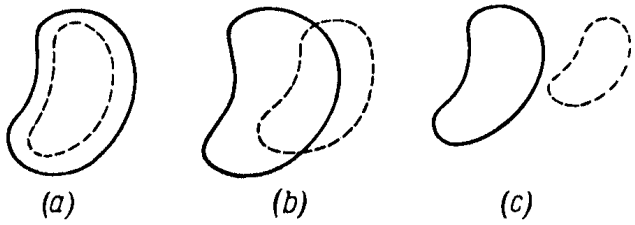

FIG. 82.

Then we can state that the body has the volume given by the formula

$$V = \int_a^b P(x)\,dx. \tag{8}$$

To prove the statement subdivide the interval $[a, b]$ of the x-axis by the points

$$a = x_0 < x_1 < \ldots < x_i < x_{i+1} < \ldots < x_n = b$$

and subdivide the body into layers by means of the planes $x = x_i$ passing through the above points. Consider the ith layer contained between the planes $x = x_i$ and $x = x_{i+1}$ $(i = 0, 1, \ldots, n-1)$. Let M_i be the greatest value and m_i the least value of the function $P(x)$ over the subinterval $[x_i, x_{i+1}]$; if the cross-sections corresponding to distinct values of x in this interval are projected onto one plane, say $x = x_i$, then, by the above assumption, they are all contained in the greatest area (M_i) and will all contain the smallest area (m_i). If on the greatest and the smallest cross-sections we construct straight cylinders with heights $\Delta x_i = x_{i+1} - x_i$, the greater contains the considered layer of the body and the smaller is itself contained in this layer; the volumes of these cylinders are $M_i \Delta x_i$ and $m_i \Delta x_i$, respectively.

The interior cylinders constitute a body (T) and the exterior a body (U), both step figures; their volumes are

$$\sum_i M_i \Delta x_i \quad \text{and} \quad \sum_i m_i \Delta x_i$$

respectively, and when $\lambda = \max \Delta x_i$ tends to zero they have the common limit (8). In view of Sec. 197, *(6)* this is the volume of the body (V)†.

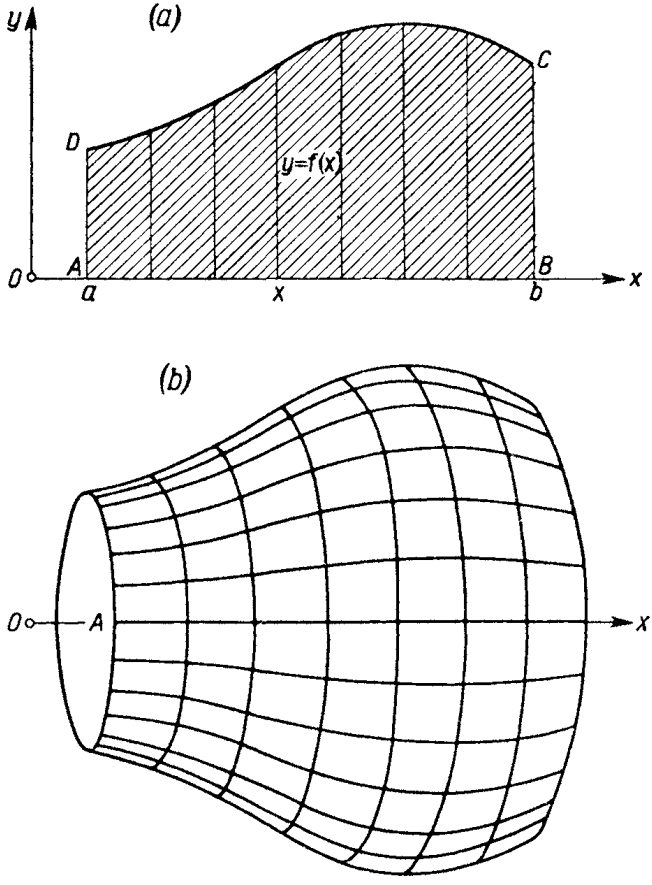

FIG. 83.

The bodies of revolution form an important particular case when the assumption concerning the mutual location of the cross-sections

† Dividing, for instance, the interval into equal parts it is easy to separate the sequences of interior and exterior bodies considered in the proposition.

is certainly satisfied. Consider a curve in the xy plane given by the equation $y = f(x)$ $(a \leqslant x \leqslant b)$ where $f(x)$ is continuous and non-negative; let us rotate the curvilinear trapezium bounded by the curve about the x-axis (Fig. 83 *a* and *b*). The body (V) so obtained is evidently the required one, since the projections of its cross-sections onto a plane perpendicular to the x-axis are concentric circles. Here

$$P(x) = \pi y^2 = \pi[f(x)]^2,$$

and hence

$$V = \pi \int_a^b y^2 dx = \pi \int_a^b [f(x)]^2 dx. \tag{9}$$

If the curvilinear trapezium is bounded both below and above by the curves $y_1 = f_1(x)$, $y_2 = f_2(x)$, then evidently

$$V = \pi \int_a^b [y_2^2 - y_1^2] dx = \pi \int_a^b \{[f_2(x)]^2 - [f_1(x)]^2\} dx, \tag{10}$$

although it may happen that the assumption concerning the cross-sections is not satisfied. In general the above result can easily be extended to all bodies which can be formed from the addition or subtraction of bodies satisfying the above assumption.

In the general case we may assert that only if the body (V) possesses a volume† it is given by formula (9).

Examples. (1) Suppose that the ellipse $x^2/a^2 + y^2/b^2 = 1$ is rotated about the x-axis. Since

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2),$$

we have the following expression for the volume of the ellipsoid of revolution:

$$V = \pi \int_{-a}^{a} \frac{b^2}{a^2}(a^2 - x^2) dx = 2\pi \frac{b^2}{a^2} \int_0^a (a^2 - x^2) dx$$

$$= 2\pi \frac{b^2}{a^2}\left(a^2 x - \frac{x^3}{3}\right)\Big|_0^a = \frac{4}{3}\pi a b^2 ‡.$$

† This is, for instance, the case if the body satisfies the conditions of *(3)*.

‡ It is readily observed that $\int_{-a}^{0} = \int_0^a$ (substituting $x = -t$).

Similarly, for the volume of a body obtained by rotation about the y-axis, we have $4\pi a^2 b/3$. Setting $a = b = r$ we obtain the familiar expression $4\pi r^3/3$ for the volume of a sphere of radius r.

(2) We now consider the branch of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$ such that $0 \leqslant t \leqslant 2\pi$ in a similar way.

By substituting the parametric equations of the curve $x = a(t - \sin t)$, and $dx = a(1 - \cos t)dt$, in the formula

$$V = \pi \int_0^{2\pi a} y^2 dx,$$

we find

$$V = \pi a^3 \int_0^{2\pi} (1 - \cos t)^3 dt = \pi a^3 \left(\frac{5}{2} t - 4 \sin t + \frac{3}{4} \sin 2t + \frac{1}{3} \sin^3 t \right) \Bigg|_0^{2\pi} = 5\pi^2 a^3.$$

(3) We now find the volume of a general ellipsoid given by the canonical equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

(Fig. 84).

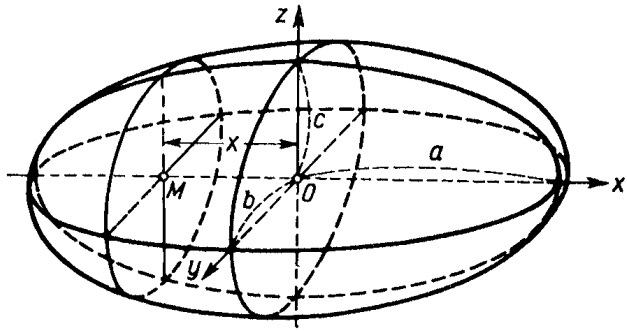

FIG. 84.

The plane perpendicular to the x-axis and passing through the point $M(x)$ of this axis intersects the ellipsoid in an ellipse; the equation of its projection (without deformation) onto the yz-plane is

$$\frac{y^2}{b^2\left(1 - \dfrac{x^2}{a^2}\right)} + \frac{z^2}{c^2\left(1 - \dfrac{x^2}{a^2}\right)} = 1 \quad (x = \text{const}).$$

It is therefore clear that its semi-axes are

$$b\sqrt{\left(1 - \frac{x^2}{a^2}\right)} \quad \text{and} \quad c\sqrt{\left(1 - \frac{x^2}{a^2}\right)},$$

respectively, and the area [Sec. 196, *(1)*] has the form

$$P(x) = \pi bc\left(1 - \frac{x^2}{a^2}\right) = \frac{\pi bc}{a^2}(a^2 - x^2).$$

Thus, by formula (8) the required volume is

$$V = \frac{\pi bc}{a^2}\int_{-a}^{a}(a^2 - x^2)dx = \frac{4}{3}\pi abc.$$

(4) Consider two circular cylinders of radius r the axes of which intersect at a right angle; we find the common volume of the two cylinders.

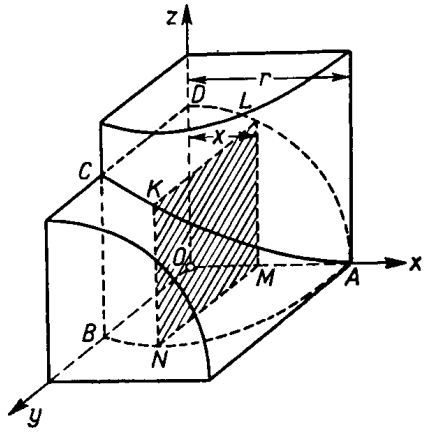

FIG. 85.

The body $OABCD$ shown in Fig. 85 is one-eighth of the considered body. The x-axis is drawn through the point O of the intersection of the axes of the cylinders, perpendicularly to their axes. Then in the cross-section (perpendicular to the x-axis) of the body $OABCD$ by a plane at distance x from O, we obtain a square $KLMN$ the side of which is $MN = \sqrt{(r^2 - x^2)}$. Hence $P(x) = r^2 - x^2$. By formula (8)

$$V = 8\int_{0}^{r}(r^2 - x^2)dx = \frac{16}{3}r^3.$$

(5) Finally we solve the same problem but now for the case when the cylinders have different radii r and $R > r$.

The only difference is that now instead of the cross-section of the considered body by a plane at distance x from O being a square, it is a rectangle with sides $\sqrt{(r^2 - x^2)}$ and $\sqrt{(R^2 - x^2)}$. Thus in this case the volume V takes the form of the elliptic integral

$$V = 8\int_{0}^{r}\sqrt{[(R^2 - x^2)(r^2 - x^2)]}dx$$

or, substituting $x = r\sin\varphi$ and setting $k = r/R$,

$$V = 8Rr^2 \int_0^{\frac{\pi}{2}} \cos^2\varphi \sqrt{(1 - k^2\sin^2\varphi)}\, d\varphi = 8Rr^2 I.$$

Let us reduce the integral I to complete elliptic integrals† (of both kinds). We have

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos^2\varphi}{\sqrt{(1 - k^2\sin^2\varphi)}}\, d\varphi - k^2 \int_0^{\frac{\pi}{2}} \frac{\sin^2\varphi\cos^2\varphi}{\sqrt{(1 - k^2\sin^2\varphi)}}\, d\varphi = I_1 + I_2.$$

But

$$I_1 = \int_0^{\frac{\pi}{2}} \frac{1 - \sin^2\varphi}{\sqrt{(1 - k^2\sin^2\varphi)}}\, d\varphi = \frac{k^2 - 1}{k^2} \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{(1 - k^2\sin^2\varphi)}}$$

$$+ \frac{1}{k^2} \int_0^{\frac{\pi}{2}} \sqrt{(1 - k^2\sin^2\varphi)}\, d\varphi = \left(1 - \frac{1}{k^2}\right) F(k) + \frac{1}{k^2} E(k).$$

On the other hand, integrating by parts

$$I_2 = \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin 2\varphi\, d\sqrt{(1 - k^2\sin^2\varphi)}$$

$$= \frac{1}{2} \sin 2\varphi \sqrt{(1 - k^2\sin^2\varphi)} \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \cos 2\varphi \sqrt{(1 - k^2\sin^2\varphi)}\, d\varphi$$

$$= \int_0^{\frac{\pi}{2}} (1 - 2\cos^2\varphi) \sqrt{(1 - k^2\sin^2\varphi)}\, d\varphi = E(k) - 2I.$$

Hence

$$I = \frac{1}{3}\left[\left(\frac{1}{k^2} + 1\right) E(k) - \left(\frac{1}{k^2} - 1\right) F(k)\right].$$

Thus, finally,

$$V = \frac{8R^3}{3} [(1 + k^2) E(k) - (1 - k^2) F(k)].$$

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† See the footnote on p. 379.

§ 2. Length of arc

199. Definition of the concept of the length of an arc. Consider a plane *open* curve AB given by the parametric equations

$$x = \varphi(t), \quad y = \psi(t), \quad (t_0 \leqslant t \leqslant T) \tag{1}$$

the functions φ and ψ being assumed to be continuous. We suppose that the points A and B correspond to the values $t = t_0$ and $t = T$, respectively. We assume that there are no multiple points on the curve and so that to two distinct values of t there correspond two distinct points of the curve.

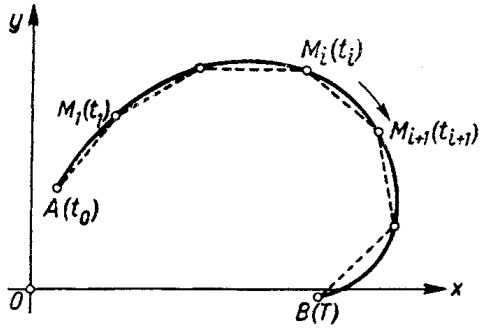

FIG. 86.

If we assume that the points of the curve are ordered with respect to increasing t (i.e. the point corresponding to the greater value of the parameter follows the point corresponding to a smaller value), then we can associate a definite direction with the curve (Fig. 86).

Now take a sequence of points

$$A = M_0, M_1, M_2, \ldots, M_i, M_{i+1}, \ldots, M_m = B$$

on the curve AB ordered in the above direction; they correspond to the increasing sequence of values of the parameter

$$t_0 < t_1 < t_2 < \ldots < t_i < t_{i+1} < \ldots < t_m = T.$$

We inscribe about the curve AB a broken line $(p) = AM_1M_2 \ldots B$ and denote its perimeter by p.

The finite limit s (provided it exists) of the perimeter p when the greatest side M_iM_{i+1} of the broken line (p) tends to zero is said to be the length of the arc

$$s = \overset{\frown}{AB} = \lim p.$$

If this limit exists the curve is said to be *rectifiable.*

The meaning of this definition can also be expressed as follows: for any sequence of broken lines $\{(p_n)\}$ inscribed about the curve (which satisfy the single condition that the greatest side of (p_n) tends to zero as n increases), the perimeter p_n always tends to the limit s.

This result can also be stated in the "ε–δ language": for any $\varepsilon > 0$ a number $\delta > 0$ can be found, such that the inequality

$$0 \leqslant s - p < \varepsilon$$

is satisfied, provided all sides of the inscribed line satisfy the inequality

$$\overline{M_iM_{i+1}} < \delta.$$

The equivalence of the two definitions is proved in the usual way.

An important property of the length of an arc is its *additivity*:

If we take a point C on the arc AB, the rectifiability of the arc AB implies the rectifiability of the two arcs AC and CB, and

$$\overset{\frown}{AB} = \overset{\frown}{AC} + \overset{\frown}{CB}.$$

We accept this statement without proof: for the curves with which we shall usually be concerned [see Sec. 201], not only the existence of the length of arc is ensured but the additivity follows from the expression of the length of the arc as an integral.

Now consider the case of a *closed* curve for which the points A and B coincide (but still there are no multiple points, i.e. every point other than $A = B$ corresponds to one only value of the parameter t). It is readily seen that in this case the above definition of the length of arc cannot immediately be applied; in fact, even if the above condition is satisfied, the broken line could reduce to a point and the perimeter to zero (Fig. 87). The essence of the problem is that for an open curve the decrease of all the chains of the broken

line (p) to zero alone ensures that the chords of (p) tend to the corresponding segments of the arc AB; hence, it is natural to take the limit of the perimeter p as the length of the whole arc. In the case of a closed curve, however, the situation is different†.

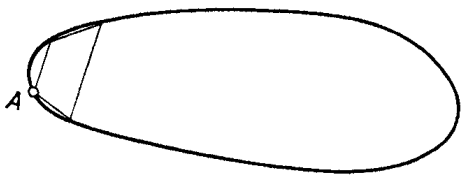

FIG. 87.

We could modify the definition (necessarily complicating it) to include the case of a closed curve. For simplicity we prefer to proceed in another way; we divide a closed curve by means of a point C on it into two open pieces and we call the sum of their lengths (if they both are rectifiable) the length of the whole curve. By the additivity of the length of the arc it can easily be proved that the sum in fact is independent of the choice of the points A and C.

200. Lemmas. Again consider an open curve (1) without multiple points. We shall prove the following two auxiliary propositions.

LEMMA 1. *If the points M' and M'' correspond to the values t' and t'' of the parameter $(t'<t'')$, then for any $\delta>0$ a number $\eta>0$ can be found, such that for $t''-t'<\eta$ the length of the chord satisfies the inequality $\overline{M'M''}<\delta$.*

In fact, by the uniform continuity of the functions φ and ψ entering (1), for a given $\delta>0$ a number $\eta>0$ can be found, such that when $|t''-t'|<\eta$ we have, simultaneously,

$$|\varphi(t'')-\varphi(t')|<\frac{\delta}{\sqrt{2}}, \quad |\psi(t'')-\psi(t')|<\frac{\delta}{\sqrt{2}},$$

and hence

$$\overline{M'M''}=\sqrt{\{[\varphi(t'')-\varphi(t')]^2+[\psi(t'')-\psi(t')]^2\}}<\delta.$$

We also have the following

LEMMA 2. *For any $\eta>0$ a number $\delta>0$ exists, such that if the length of the chord $\overline{M'M''}<\delta$ the difference $t''-t'$ $(t'<t'')$ of the values of the parameter corresponding to its end-points is smaller than η.*

---

† Recalling from school courses of elementary geometry the definition of the length of circumference as the limit of the perimeter of the inscribed regular polygon, we find that the assumption on the regularity of the polygon eliminates the above possibility.

Assume the converse; then for some $\eta > 0$, and for any $\delta > 0$, two points $M'(t')$ and $M''(t'')$ can be found such that $\overline{M'M''} < \delta$ and $t'' - t' \geqslant \eta$. Taking the sequence $\{\delta_n\}$ converging to zero we arrive at two sequences of points $\{M_n'(t_n')\}$ and $\{M_n''(t_n'')\}$ for which

$$\overline{M_n' M_n''} < \delta_n, \quad \text{but} \quad t_n'' - t_n' \geqslant \eta \quad (n = 1, 2, 3, \ldots).$$

By the Bolzano–Weierstrass lemma [Sec. 51] we may assume, without loss of generality, that

$$t_n' \to t^*, \quad t_n'' \to t^{**}$$

(this can easily be achieved by considering, if necessary, subsequences). Obviously

$$t^{**} - t^* \geqslant \eta,$$

and hence $t^* \neq t^{**}$. At the same time, for the corresponding points M^* and M^{**} we have $\overline{M^* M^{**}} = 0$, i.e. these points should coincide, which is impossible since the curve has no multiple points and is open. This contradiction proves the statement.

The above two lemmas indicate that in the definition of the length of an open curve it is entirely irrelevant whether we require that the greatest side of the inscribed line tends to zero [by Sec. 199], or that we require that the greatest difference $\Delta t_i = t_{i+1} - t_i$ tends to zero; in fact, these requirements are equivalent. It will now be convenient to employ the latter condition.

201. Integral expression for the length of an arc. We now assume, in addition, that the functions φ and ψ appearing in (1), for an *open* curve, have continuous derivatives φ' and ψ'. We now prove that under these conditions the curve is rectifiable and the length of the arc is given by the formula

$$s = \int_{t_0}^{T} \sqrt{(x_t'^2 + y_t'^2)}\, dt = \int_{t_0}^{T} \sqrt{\{[\varphi'(t)]^2 + [\psi'(t)]^2\}}\, dt. \tag{2}$$

We subdivide the interval $[t_0, T]$ by means of the points

$$t_0 < t_1 < t_2 < \ldots < t_i < t_{i+1} < \ldots < t_n = T$$

into parts of lengths $\Delta t_i = t_{i+1} - t_i$. To these values of t correspond the vertices of the broken line $AM_1 \ldots M_{n-1}B$ inscribed in arc $\overset{\frown}{AB}$ and (as we have shown above) its length s may be defined as the limit of the perimeter p of the broken line when $\lambda^* = \max \Delta t_i$ tends to zero.

Set

$$\varphi(t_i) = x_i, \quad \psi(t_i) = y_i \quad (i = 0, 1, \ldots, n)$$

and

$$\Delta x_i = x_{i+1} - x_i, \quad \Delta y_i = y_{i+1} - y_i \quad (i = 0, 1, \ldots, n-1).$$

The length of ith chord M_iM_{i+1} of the inscribed line has the form

$$\overline{M_iM_{i+1}} = \sqrt{(\Delta x_i^2 + \Delta y_i^2)}.$$

The formula of finite increments applied to the increments Δx_i and Δy_i of function (1) gives

$$\Delta x_i = \varphi(t_i + \Delta t_i) - \varphi(t_i) = \varphi'(\tau_i)\Delta t_i,$$
$$\Delta y_i = \psi(t_i + \Delta t_i) - \psi(t_i) = \psi'(\tau_i^*)\Delta t_i,$$

where we know nothing about the values τ_i and τ_i^*, except that they lie between t_i and t_{i+1}. Hence we have

$$\overline{M_iM_{i+1}} = \sqrt{\{[\varphi'(\tau_i)]^2 + [\psi'(\tau_i^*)]^2\}}\Delta t_i,$$

and we obtain the expression

$$p = \sum_i \sqrt{\{[\varphi'(\tau_i)]^2 + [\psi'(\tau_i^*)]^2\}}\Delta t_i$$

for the perimeter of the broken line.

If we replace τ_i^* by τ_i in the second term under the root the resultant expression

$$\sigma = \sum_i \sqrt{\{[\varphi'(\tau_i)]^2 + [\psi'(\tau_i)]^2\}}\Delta t_i$$

evidently represents the integral sum of integral (2). When λ^* tends to zero the above integral is the limit of this† sum. To prove that this is also the limit of the perimeter p of the broken line it is sufficient to prove that the difference $p - \sigma$ tends to zero.

For this purpose we estimate this difference

$$|p - \sigma| \leqslant \sum_i |\sqrt{\{[\varphi'(\tau_i)]^2 + [\psi'(\tau_i^*)]^2\}} - \sqrt{\{[\varphi'(\tau_i)]^2 + [\psi'(\tau_i)]^2\}}|\Delta t_i.$$

If we apply the elementary inequality

$$|\sqrt{(a^2 + b_1^2)} - \sqrt{(a^2 + b^2)}| \leqslant |b_1 - b|\ddagger$$

† Its existence is obvious since the integrand is continuous [Sec. 179, I].

‡ This inequality is evident for $a = 0$; if $a \neq 0$ it follows directly from the identity

$$\sqrt{(a^2 + b_1^2)} - \sqrt{(a^2 + b^2)} = \frac{b_1 + b}{\sqrt{(a^2 + b_1^2)} + \sqrt{(a^2 + b^2)}}(b_1 - b),$$

since the absolute value of the factor of the difference $(b_1 - b)$ is smaller than unity.

to every term of the above sum, separately, we obtain

$$|p-\sigma| \leqslant \sum_i |\psi'(\tau_i^*)-\psi'(\tau_i)|\Delta t_i.$$

By the continuity of the function $\psi'(t)$, for any given $\varepsilon>0$ a number $\delta>0$ can be found such that $|\psi'(t^*)-\psi'(t)|<\varepsilon$ provided $|t^*-t|<\delta$. If we take all $\Delta t_i<\delta$, then $|\tau_i^*-\tau_i|<\delta$ and, hence, $|\psi'(\tau_i^*)-\psi'(\tau_i)|<\varepsilon$; moreover,

$$|p-\sigma| \leqslant \varepsilon \sum_i \Delta t_i = \varepsilon(T-t_0).$$

This completes the proof.

If the curve is given by an explicit equation in rectangular coordinates

$$y=f(x) \qquad (x_0 \leqslant x \leqslant X);$$

taking x as the parameter, formula (2) gives as a particular case

$$s=\int_{x_0}^{X} \sqrt{(1+y_x'^2)}\,dx = \int_{x_0}^{X} \sqrt{\{1+[f'(x)]^2\}}\,dx. \tag{2a}$$

Finally, if the curve is given by a polar equation

$$r=g(\theta) \qquad (\theta_0 \leqslant \theta \leqslant \Theta)$$

we can again obtain a parametric representation by means of the usual formulae

$$x=r\cos\theta=g(\theta)\cos\theta, \qquad y=r\sin\theta=g(\theta)\sin\theta,$$

where θ is the parameter. Now we have

$$x_\theta'=r_\theta'\cos\theta-r\sin\theta, \qquad y_\theta'=r_\theta'\sin\theta+r\cos\theta,$$

hence

$$x_\theta'^2+y_\theta'^2=r^2+r_\theta'^2 \tag{3}$$

and

$$s=\int_{\theta_0}^{\Theta} \sqrt{(r^2+r_\theta'^2)}\,d\theta = \int_{\theta_0}^{\Theta} \sqrt{\{[g(\theta)]^2+[g'(\theta)]^2\}}\,d\theta. \tag{2b}$$

Remark. Formula (2) can be extended directly to the case of a closed curve. In this case take an arbitrary t' between t_0 and T and divide the closed curve (1) by the corresponding point $M'(t')$

into two open curves AM' and $M'B$ and apply to each new curve separately the formula of type (2)

$$s_1 = \overset{\frown}{AM'} = \int_{t_0}^{t'}, \qquad s_2 = \overset{\frown}{M'B} = \int_{t'}^{T}.$$

Adding the results, we obtain for the whole closed curve

$$s = s_1 + s_2 = \int_{t_0}^{T}.$$

Examples. (1) The parabola $y = x^2/2p$. Measuring the length from the vertex O ($x = 0$) we have for an arbitrary point M with abscissa x

$$\begin{aligned} s = \overset{\frown}{OM} &= \frac{1}{p}\int_0^x \sqrt{(x^2+p^2)}\,dx \\ &= \frac{1}{p}\left[\frac{1}{2}x\sqrt{(x^2+p^2)} + \frac{p^2}{2}\log\left[x+\sqrt{(x^2+p^2)}\right]\right]_0^x \\ &= \frac{x}{2p}\sqrt{(x^2+p^2)} + \frac{p}{2}\log\frac{x+\sqrt{(x^2+p^2)}}{p}. \end{aligned}$$

(2) The cycloid $x = a(t-\sin t)$, $y = a(1-\cos t)$.

Here (for $0 \leqslant t \leqslant 2\pi$)

$$\sqrt{(x_t'^2 + y_t'^2)} = a\sqrt{[(1-\cos t)^2 + \sin^2 t]} = 2a\sin\frac{t}{2};$$

by formula (2) the length of one branch of the cycloid is

$$S = 2a\int_0^{2\pi}\sin\frac{t}{2}\,dt = -4a\cos\frac{t}{2}\Big|_0^{2\pi} = 8a.$$

(3) The Archimedean spiral $r = a\theta$.

By formula (2b), measuring the arc from the point O to any point M (corresponding to the angle θ), we have

$$s = \overset{\frown}{OM} = a\int_0^{\theta}\sqrt{(1+\theta^2)}\,d\theta = \frac{a}{2}\{\theta\sqrt{(1+\theta^2)} + \log[\theta+\sqrt{(1+\theta^2)}]\}.$$

It is interesting to note that, substituting $\theta = r/a$, we arrive at an expression which is formally similar to the expression for the length of arc of a parabola (see *(1)*).

(4) The ellipse $x^2/a^2 + y^2/b^2 = 1$.

It is more convenient to take the equation of the ellipse in the parametric form $x = a\sin t$, $y = b\cos t$. Obviously

$$\begin{aligned} \sqrt{(x_t'^2 + y_t'^2)} &= \sqrt{(a^2\cos^2 t + b^2\sin^2 t)} = \sqrt{[a^2 - (a^2-b^2)\sin^2 t]} \\ &= a\sqrt{(1-\varepsilon^2\sin^2 t)} \end{aligned}$$

where $\varepsilon = \sqrt{(a^2-b^2)}/a$ is the eccentricity of the ellipse.

Calculating the length of arc of an ellipse from the upper end of the minor axis to an arbitrary point of t in the first quadrant, we obtain

$$s = a\int_0^t \sqrt{(1-\varepsilon^2\sin^2 t)}\,dt = aE(\varepsilon, t).$$

Thus the length of arc of an ellipse is given by an elliptic integral of the second kind [Secs. 174, 183]; it has already been indicated that this was the reason for the name "elliptic integral".

In particular, the length of a quarter of the perimeter of the ellipse is expressed by the complete elliptic integral†

$$a\int_0^{\frac{\pi}{2}} \sqrt{(1-\varepsilon^2\sin^2 t)}\,dt = aE(\varepsilon).$$

The length of the whole perimeter is

$$S = 4aE(\varepsilon).$$

202. Variable arc and its differential. Let the point M on the arc AB correspond to an arbitrary value of t. Then the length of the arc $\overset{\frown}{AM}$ is expressed by the formula

$$s = s(t) = \overset{\frown}{AM} = \int_{t_0}^{t} \sqrt{(x_t'^2 + y_t'^2)}\,dt \tag{4}$$

instead of (2). Evidently it is an increasing continuous function of t.

Moreover, by the continuity of the integrand the variable arc $s = s(t)$ has a derivative with respect to t equal to the integrand [Sec. 183, *(12)*]:

$$s_t' = \sqrt{(x_t'^2 + y_t'^2)}. \tag{5}$$

Squaring and multiplying by dt^2, we arrive at the remarkably simple formula

$$ds^2 = dx^2 + dy^2, \tag{6}$$

which, moreover, has a clear geometric interpretation. In Fig. 88 in the curvilinear rectangular polygon MNM_1 the sides adjacent to the right angle are the increments of the coordinates of the point M: $MN = \Delta x$, $NM_1 = \Delta y$ and the "hypotenuse" is the arc $\overset{\frown}{MM_1} = \Delta s$, which is the increment of the arc $\overset{\frown}{AM} = s$. It turns

† See the footnote on p. 379.

out that at least for the differentials of the increments, if not for the increments themselves, we have a special "Pythagoras theorem".

It is useful to note particular cases of the important formula (5), corresponding to various particular types of representation of the curve. Thus, if the curve is given by an explicit equation in Cartesian coordinates $y = f(x)$, then the role of the parameter is played by x and the arc function is $s = s(x)$. Formula (5) takes the form

$$s'_x = \sqrt{(1 + y'^2_x)}. \tag{5a}$$

If the curve is represented by the polar equation $r = g(\theta)$ and the parameter is θ the arc now is a function of θ: $s = s(\theta)$. In view of (3), formula (5) takes the form

$$s'_\theta = \sqrt{(r^2 + r'^2_\theta)}. \tag{5b}$$

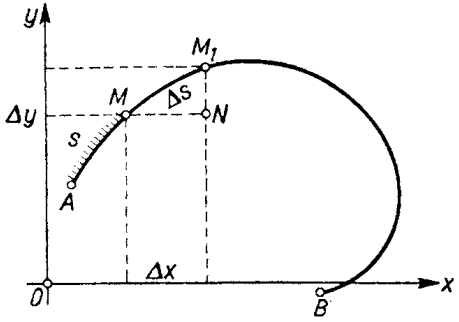

FIG. 88.

It is frequently convenient to take as the initial point A from which the length of the arc is measured, not one of the ends of the arc but some interior point. In this case it is natural to regard the arc lengths in the direction of increasing parameter as positive, and those counted in the opposite direction as negative; accordingly, in the first case the length of the arc has the positive sign and in the second case negative. This value of the arc, with its sign, we shall call, for brevity, simply the *arc*. Formulae (4), (5), (5a), (5b) hold in all cases.

Since the variable arc $s = \overset{\frown}{AM}$ is a continuous monotonically increasing function of the parameter t the latter can be regarded as a single-valued and continuous function of s: $t = \omega(s)$ [Sec. 71].

Substituting this expression into the equations (1) we obtain the coordinates x and y as functions of s

$$x = \varphi(\omega(s)) = \Phi(s), \quad y = \psi(\omega(s)) = \Psi(s).$$

Clearly the arc $s = \overset{\frown}{AM}$, regarded as a "curvilinear abscissa" of the point M, is itself a natural parameter for the determination of the location of M.

Assume that for a given value of t the two derivatives x_t' and y_t' do not simultaneously vanish (the geometric meaning of this assumption will be explained in Sec. 210); then

$$s_t' = \sqrt{(x_t'^2 + y_t'^2)} > 0,$$

and for the corresponding value of s the derivative [Sec. 80]

$$t_s' = \omega'(s) = \frac{1}{\sqrt{(x_t'^2 + y_t'^2)}}$$

exists, and consequently also the derivatives

$$x_s' = \Phi'(s), \quad y_s' = \Psi'(s).$$

203. Length of the arc of a spatial curve. For the spatial curve

$$x = \varphi(t), \quad y = \psi(t), \quad z = \chi(t),$$

without multiple points, the definition of the length of an arc may be given in the same form as for a plane curve [Secs. 199–201]. For the length of arc we obtain a formula analogous to (2),

$$s = \overset{\frown}{AB} = \int_{t_0}^{T} \sqrt{(x_t'^2 + y_t'^2 + z_t'^2)}\, dt$$

and so on. All results concerning the case of a plane curve can be extended to the case of a spatial curve almost without alterations. Without going into details, we present some examples.

(1) The circular helix: $x = a\cos t$, $y = a\sin t$, $z = ct$.
Since here

$$\sqrt{(x_t'^2 + y_t'^2 + z_t'^2)} = \sqrt{(a^2 + c^2)},$$

the length of the curve from the points A $(t = 0)$ to the point M (where t is arbitrary) is given by the formula

$$s = \overset{\frown}{AM} = \int_0^t \sqrt{(a^2 + c^2)}\, dt = \sqrt{(a^2 + c^2)}\, t;$$

the result is obvious if we remember that in developing a cylindrical surface the helix on it becomes a straight line inclined to the axis.

(2) The curve: $x = R\sin^2 t$, $y = R\sin t\cos t$, $z = R\cos t$, where $0 \leqslant t \leqslant \pi/2$. We have

$$\sqrt{(x_t'^2 + y_t'^2 + z_t'^2)} = R\sqrt{(1+\sin^2 t)}.$$

In this case the length of the whole curve is given by the complete elliptic integral of the second kind

$$S = R\int_0^{\frac{\pi}{2}} \sqrt{(1+\sin^2 t)}\,dt = R\int_0^{\frac{\pi}{2}} \sqrt{(1+\cos^2 t)}\,dt$$

$$= \sqrt{2}R\int_0^{\frac{\pi}{2}} \sqrt{\left(1-\frac{1}{2}\sin^2 t\right)}\,dt = \sqrt{2}RE\left(\frac{1}{\sqrt{2}}\right).$$

§ 3. Computation of mechanical and physical quantities

204. Applications of definite integrals. Before proceeding to applications of define integrals in the field of mechanics, physics and engineering it is first useful to examine the way in which applications usually lead to a definite integral. For this purpose we make a general plan of the application of the integral, illustrating it by examples of already investigated geometric problems.

Suppose that it is required to determine a constant quantity Q (geometric or otherwise) connected with the interval $[a, b]$. Moreover, to every subinterval $[\alpha, \beta]$ contained in $[a, b]$ let there correspond a part of the quantity Q so that the subdivision of $[a, b]$ into subintervals results in a corresponding subdivision of the quantity Q.

More precisely, we consider a "function of the interval" $Q([\alpha, \beta])$ possessing the property of additivity, so, if the interval $[\alpha, \beta]$ is split into the subintervals $[\alpha, \gamma]$ and $[\gamma, \beta]$, then

$$Q([\alpha, \beta]) = Q([\alpha, \gamma]) + Q([\gamma, \beta]).$$

The problem consists of the calculation of its value for the whole interval $[a, b]$.

For instance, consider a plane curve $y = f(x)$ $(a \leqslant x \leqslant b)$ (Fig. 89). Then *(1)* the length S of the curve AB, *(2)* the area P of the curvilinear trapezium $AA'B'B$ bounded by it, and *(3)* the volume V of the body

obtained by rotating the trapezium around the x-axis, are all quantities of the above type. It is easy to find the "functions of the interval" generated by them.

Consider an "element" ΔQ of the quantity Q corresponding to the "elementary interval" $[x, x+\Delta x]$. Under the conditions of the problem we attempt to find an approximate expression for ΔQ

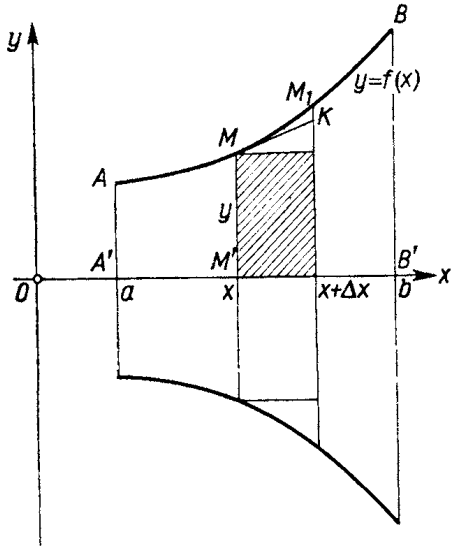

FIG. 89.

of the form $q(x)\Delta x$, linear in Δx, and which differs from ΔQ by at most an infinitesimal of a higher order than Δx. In other words, we separate the principal part from the infinitesimal (as $\Delta x \to 0$) "element". It is clear that the relative error of the approximate relation

$$\Delta Q \doteq q(x)\Delta x \tag{1}$$

tends to zero as $\Delta x \to 0$.

Thus in Example *(1)* the element $\widetilde{MM_1}$ of the arc can be replaced by a segment of the tangent MK so that the linear part

$$\sqrt{(1+y_x'^2)}\Delta x = \sqrt{\{1+[f'(x)]^2\}}\Delta x$$

is separated from ΔS. In Example *(2)* it is natural to replace the elementary strip ΔP by the interior rectangle with the area

$$y\Delta x = f(x)\Delta x.$$

Finally, in Example *(3)* we separate from the elementary layer the principal part in the form of an interior circular cylinder with the volume

$$\pi y^2 \Delta x = \pi [f(x)]^2 \Delta x.$$

In all three cases it is easy to prove that the error of such replacement is an infinitesimal of an order higher than Δx.

This being done we may state that the required quantity Q is *exactly* represented by the integral

$$Q = \int_a^b q(x)\,dx. \tag{2}$$

To elucidate the statement subdivide the interval $[a, b]$ by means of the points $x_1, x_2, \ldots, x_{n-1}$ into elementary intervals

$$[a, x_1], \quad [x_1, x_2], \ \ldots, \ [x_i, x_{i+1}], \ \ldots, \ [x_{n-1}, b].$$

Since to every interval $[x_i, x_{i+1}]$, or $[x_i, x_i + \Delta x_i]$, there corresponds an elementary part of our quantity equal approximately to $q(x_i)\,\Delta x_i$, the unknown quantity Q is approximately given by the sum

$$Q \doteq \sum_i q(x_i) \Delta x_i.$$

The smaller the subintervals, the greater is the degree of accuracy of the derived result, and consequently it is evident that Q is the limit of the sum, i.e. it is, in fact, expressed by the definite integral

$$\int_a^b q(x)\,dx.$$

This fully concerns all three considered examples. Previously we derived the formulae for S, P, V in a somewhat different way, because our task was not only to calculate them but also to prove their existence, in accordance with the established definitions.

Thus the problem is now reduced to establishing the approximate relation (1), which is usually written in the form

$$dQ = q(x)\,dx. \tag{3}$$

It remains only to "sum" these "elements", which leads to formula (2).

We emphasize that the integral must be used instead of the ordinary sum. The sum would only give an approximate expression for Q

since the error of the relations of type (3) would affect it; however, the passage to the limit which changes the sum to the integral, eliminates the error and gives an entirely exact result. Thus, in the expression for the element dQ we first disregard the infinitesimals of higher orders and we separate out the principal part; then, for the sake of exactness, the summation sign is replaced by the integral sign and the result derived in this simple way turns out to be *exact.*

Incidentally, the problem could be tackled from a different point of view. Denote by $Q(x)$ the variable part of the quantity Q which corresponds to the interval $[a, x]$, it being assumed that $Q(a)$ vanishes. Evidently the foregoing "function of an interval" $Q([\alpha, \beta])$ is expressed in terms of the "function of a point" $Q(x)$ by the relation

$$Q([\alpha, \beta]) = Q(\beta) - Q(\alpha).$$

In our examples the functions of a point are the following: *(1)* the variable arc $\overset{\frown}{AM}$, *(2)* the area of the variable trapezium $AA'M'M$ and finally *(3)* the volume of the body obtained by rotating the above trapezium.

The quantity ΔQ is simply the increment of the function $Q(x)$ and the product $q(x)dx$ represents its principal part, i.e. the differential of the function. Thus, relation (3) written in the notation of differentials is in fact not approximate but exact. This at once leads to the required result:

$$\int_a^b q(x)dx = Q(b) - Q(a) = Q([a, b]) = Q.$$

Observe, however, that in applications it is more convenient and effective to use the concept of summing infinitesimal elements (Leibniz) and then passing to the limit.

205. The area of a surface of revolution. As the first example of an application of the above plan consider the geometric problem of calculating the area of a surface of revolution.

We are not in a position to establish here the general form of the concept of the area of a curved (i.e. not plane) surface; this will be done in the second volume. Therefore we confine ourselves to finding the area of a surface of revolution assuming that it exists and possesses the property of additivity. We shall subsequently

find that the deduced formula is a particular case of a more general formula for the area of a curved surface.

Thus, consider on the xy-plane (in the upper semi-plane) a curve AB given by the equations

$$x = \varphi(t), \quad y = \psi(t), \quad (t_0 \leqslant t \leqslant T) \tag{4}$$

where φ and ψ are functions of a parameter; together with their derivatives they are assumed to be continuous. For simplicity we assume that the curve is open and has no multiple points.

In this case it is convenient to take the arc s measured from a point $A(t_0)$ as the parameter and to use the representation

$$x = \Phi(s), \quad y = \Psi(s) \quad (0 \leqslant s \leqslant S) \tag{5}$$

considered in Sec. 202. The parameter s varies from 0 to S, the latter symbol denoting the length of the curve AB.

The problem consists in determining the area Q of the surface obtained by rotating the curve AB around the x-axis. We draw the reader's attention to the fact that s plays the role of the variable, the interval of its variation being $[0, S]$.

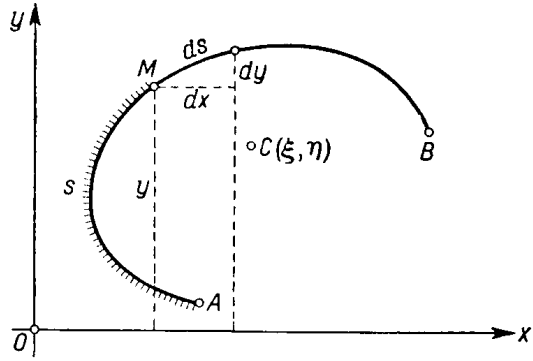

FIG. 90.

If we consider the element ds of the curve (Fig. 90), it can be approximately regarded as rectilinear and we may calculate the corresponding element of area dQ as the area of the truncated cone with the generator ds and base of radii y and $y + dy$. Then, by a formula known from school courses, we have

$$dQ = 2\pi \frac{y + (y + dy)}{2} ds.$$

This is not yet the formula we require; in fact, the product $dy \cdot ds$ of two infinitesimals must be disregarded. We arrive at the following formula linear in ds:

$$dQ = 2\pi y\, ds;$$

hence, "summing", we finally obtain

$$Q = 2\pi \int_0^S y\, ds, \tag{6}$$

where by y we understand the function $\Psi(s)$ of (5).

Returning to the general parametric representation (4) of our curve, changing the variable in the integral [Sec. 186, (2)], we obtain

$$Q = 2\pi \int_{t_0}^{T} y \sqrt{(x_t'^2 + y_t'^2)}\, dt = 2\pi \int_{t_0}^{T} \psi(t) \sqrt{\{[\varphi'(t)]^2 + [\psi'(t)]^2\}}\, dt. \tag{6a}$$

In particular, if the curve is given by the explicit equation $y = f(x)$ $(a \leqslant x \leqslant b)$ so that x is the parameter, we have

$$Q = 2\pi \int_a^b y \sqrt{(1 + y_x'^2)}\, dx = 2x \int_a^b f(x) \sqrt{\{1 + [f'(x)]^2\}}\, dx. \tag{6b}$$

Examples. (1) To calculate the area of the surface of a spherical strip.

Let the semicircle with centre at the origin and radius r be rotated around the x-axis. From the equation of the semicircle we have $y = \sqrt{(r^2 - x^2)}$ and furthermore,

$$y_x' = -\frac{x}{\sqrt{(r^2 - x^2)}}, \quad \sqrt{(1 + y_x'^2)} = \frac{r}{\sqrt{(r^2 - x^2)}},$$

$$y \sqrt{(1 + y_x'^2)} = r.$$

Thus the area of the surface of the strip described by the arc whose ends have the abscissae x_1 and $x_2 > x_1$ is, by formula (6b),

$$Q = 2\pi \int_{x_1}^{x_2} r\, dx = 2\pi r(x_2 - x_1) = 2\pi r h,$$

h being the height of the strip. Thus the area of the surface of a spherical strip is equal to the product of the circumference of the large circle and the height of the strip.

In particular, if $x_1 = -r$, $x_2 = r$, i.e. when $h = 2r$ we obtain the area of the whole spherical surface: $Q = 4\pi r^2$.

(2) To calculate the area of the surface generated by revolving an arc of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$.

Since $y = 2a \sin^2(t/2)$, $ds = 4a \sin (t/2)\, dt$ we have

$$Q = 2\pi \int_0^{2\pi} 4a^2 \sin^3 \frac{t}{2}\, dt = 16\pi a^2 \int_0^{\pi} \sin^3 u\, du$$

$$= 16\pi a^2 \left(\frac{\cos^3 u}{3} - \cos u \right) \Big|_0^{\pi} = \frac{64}{3} \pi a^2.$$

206. Calculation of static moments and centre of mass of a curve. It is known that the static moment K of a particle of mass m about an axis is equal to the product of the mass m and the distance d of the point from the axis. In the case of a system of particles with masses $m_1, m_2, \ldots, m_n$ lying in a plane at distances from the axis $d_1, d_2, \ldots, d_n$, respectively, the static moment is given by the sum

$$K = \sum_i m_i d_i.$$

The distances of points located on one side of the axis are taken with the positive sign and those on the other side with the negative sign.

If the masses, instead of being concentrated at separate points, are distributed in a continuous manner over a line or a plane figure, then to express the static moment we use an integral instead of a sum.

Let us determine the static moment K about the x-axis of masses distributed along a plane curve AB (Fig. 90). We assume that the curve is homogeneous and hence the linear density ϱ (i.e. the mass per unit length) is constant; for simplicity we assume also that $\varrho = 1$ (otherwise the result we derive has to be multiplied by ϱ). Under these assumptions the mass of an arbitrary arc of the considered curve is measured simply by its length and the concept of the static moment acquires a purely geometric character. Observe that, in general, when we speak of a static moment (or centre of mass) of a curve without mentioning the distribution of mass along it, we shall always mean the static moment (centre of mass) defined under the above assumptions.

Again consider an element ds of the curve (the mass is also given by the number ds). Approximately regarding this element as a particle at a distance y from the axis we obtain for its static moment the expression

$$dK_x = y\, ds.$$

Summing these elementary static moments and taking the independent variable as the arc s measured from a point A, we obtain

$$K_x = \int_0^S y\,ds.$$

An analogous expression is obtained for the moment about the y-axis

$$K_y = \int_0^S x\,ds.$$

Obviously we assumed that y (or x) can be expressed in terms of s. In practice, in these formulae s is expressed in terms of one of the variables t, x or θ—whichever is the independent variable in the analytic representation of the curve.

Knowing the static moments K_x and K_y of the curve we can easily determine the centre of mass $C(\xi, \eta)$ of the curve. The point C has the property that if the whole "mass" (expressed by the same number as the length) be concentrated at this point, the moment of this mass about an arbitrary axis is equal to the moment of the curve about the same axis; in particular, if we consider moments about the coordinate axes we obtain

$$S\xi = K_y = \int_0^S x\,ds, \qquad S\eta = K_x = \int_0^S y\,ds,$$

whence

$$\xi = \frac{K_y}{S} = \frac{\int_0^S x\,ds}{S}, \qquad \eta = \frac{K_x}{S} = \frac{\int_0^S y\,ds}{S}. \tag{7}$$

From the formula for the ordinate η of the centre of mass we infer a remarkable geometric result. In fact, we have

$$\eta S = \int_0^S y\,ds,$$

and hence

$$2\pi\eta \cdot S = 2\pi \int_0^S y\,ds;$$

but the right-hand side of the last relation is the area Q of the surface obtained by revolving the curve AB [Sec. 205, (6)], while on the left-hand side $2\pi\eta$ represents the length of the circumference described by the centre of mass of the curve when rotated around the x-axis, and S is the length of the considered curve. Thus we arrive at the following theorem of Guldin†.

The value of the area of the surface obtained by revolving a curve around an axis which does not intersect it is equal to the length of this curve multiplied by the length of the circumference of the circle described by the centre of mass C of the curve (Fig. 90).

This theorem enables us to determine the coordinate η of the centre of mass of the curve if the length S of the path described by it and the area Q of the surface of revolution, are known. We give some examples.

(1) Making use of Guldin's theorem determine the position of the centre of mass of the arc AB of a circle of radius r (Fig. 91).

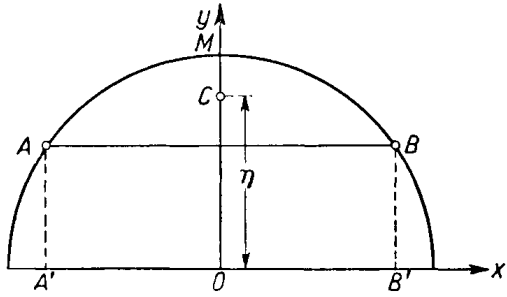

FIG. 91.

Since the arc is symmetric with respect to the radius OM passing through its midpoint M, the centre of mass C lies on this radius, and to completely determine its position it is only necessary to find its distance η from the centre O. We select the axes as shown in the figure, and denote the length of the arc $\overset{\frown}{AB}$ by s and the length of its chord $AB(=A'B')$ by h. Revolving the arc around the x-axis we obtain a spherical strip the area of the surface of which is known to be [Sec. 205, *(1)*] $Q = 2\pi rh$. By Guldin's theorem this area is equal to $2\pi\eta s$, so $s\eta = rh$ and $\eta = rh/s$.

† Paul Guldin (1577–1643)—a Swiss mathematician. Incidentally, both his theorems (see the next section) were known to Pappus—an outstanding Greek mathematician of the third century.

In particular, for a semicircle $h = 2r$, $s = \pi r$ and $\eta = 2r/\pi \doteq 0.637r$.

(2) To determine the centre of mass of the branch of a cycloid (Fig. 79, p. 389):

$$x = a(t - \sin t), \quad y = a(1 - \cos t) \quad (0 \leqslant t \leqslant 2\pi).$$

By symmetry it is at once clear that $\xi = \pi a$. In view of the results of Example *(2)* of Sec. 205 we obtain $\eta = 4a/3$.

207. Determination of static moments and centre of mass of a plane figure. Consider a plane figure $AA'B'B$ (Fig. 92) bounded above by the curve AB which is given by the explicit equation $y = f(x)$. Suppose there is a uniform distribution of mass over the figure and that the surface density ϱ (i.e. the mass per unit area) is constant. Without loss of generality we may assume that $\varrho = 1$, i.e. the mass of any part of our figure is measured by its area. This is always tacitly assumed when we speak simply about static moments (or centre of mass) of a plane figure.

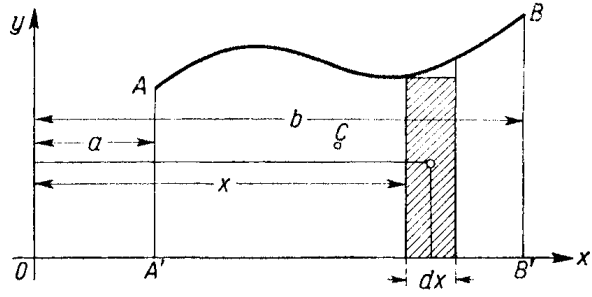

FIG. 92.

To determine the static moments K_x and K_y of our figure about the coordinate axes we consider, as before, an element of the figure in the form of a narrow vertical strip (see Fig. 92). Regarding this strip as approximately a rectangle we observe that its mass (expressed by the same number as the area) is $y\,dx$. To calculate the corresponding elementary moments dK_x and dK_y we assume that the whole mass of the strip is concentrated at the centre of mass C (i.e. at the centre of the rectangle), which, as is well known, does not influence the value of the static moments. The considered particle is at a distance $y/2$ from the x-axis and at distance $x + dx/2$ from the y-axis; the latter expression can simply be replaced by x since the

disregarded part $dx/2$ multiplied by the mass $y\,dx$ gives rise to an infinitesimal of a higher order. Thus we have

$$dK_x = \frac{1}{2}y^2\,dx, \quad dK_y = xy\,dx.$$

Summing the elementary moments we obtain

$$K_x = \frac{1}{2}\int_v^b y^2\,dx, \quad K_y = \int_a^b xy\,dx, \tag{8}$$

where y is the function $f(x)$ of the equation of the curve AB.

As in the case of a curve, knowing the static moments of the figure with respect to the coordinate axes we can easily determine the coordinates ξ, η of the centre of mass. Denoting by P the area (and consequently the mass) of the figure, by the basic property of the centre of mass

$$P\xi = K_y = \int_a^b xy\,dx, \quad P\eta = K_x = \frac{1}{2}\int_a^b y^2\,dx,$$

whence

$$\xi = \frac{K_y}{P} = \frac{\int_a^b xy\,dx}{P}, \quad \eta = \frac{K_x}{P} = \frac{\frac{1}{2}\int_a^b y^2\,dx}{P}. \tag{9}$$

In this case, too, we obtain an important geometric result from the formula for the ordinate η of the centre of mass. In fact, we have

$$2\pi\eta P = \pi\int_a^b y^2\,dx.$$

The right-hand side of this relation is the volume V of the body obtained by rotating the plane figure $AA'B'B$ around the x-axis [Sec. 198, (9)], while the left-hand side expresses the product of the area P of this figure and the length $2\pi\eta$ of the circumference described by the centre of mass of the figure. This implies Guldin's second theorem.

The volume of a body of revolution of a plane figure around an axis which does not intersect it is equal to the product of the area of

the figure and the length of the circumference described by the centre of mass of the figure:

$$V = P \cdot 2\pi\eta.$$

Observe that formulae (8) and (9) can be extended to the case of a figure bounded by curves above and below (Fig. 75, p. 387). For instance, in this case

$$K_x = \frac{1}{2}\int_a^b (y_2^2 - y_1^2)\,dx, \quad K_y = \int_a^b x(y_2 - y_1)\,dx; \tag{8a}$$

it is now clear how formulae (9) are transformed. Bearing in mind formula (5) of Sec. 196 we easily observe that Guldin's theorem holds for this case as well.

Examples. (1) To determine the static moments K_x, K_y and the coordinates of the centre of mass of a figure bounded by the parabola $y^2 = 2px$, the x-axis and the ordinate corresponding to the abscissa x.

Since $y = \sqrt{(2px)}$, by formulae (8) we have

$$K_x = \frac{1}{2}\cdot 2p\int_0^x x\,dx = \frac{1}{2}px^2, \quad K_y = \sqrt{(2p)}\int_0^x x^{\frac{3}{2}}\,dx = \frac{2\sqrt{(2p)}}{5}x^{\frac{5}{2}}.$$

On the other hand, the area has the value [Sec. 196, (4)]

$$P = \sqrt{(2p)}\int_0^x x^{\frac{1}{2}}\,dx = \frac{2\sqrt{(2p)}}{3}x^{\frac{3}{2}}.$$

Thus, by formulae (9)

$$\xi = \frac{3}{5}x, \quad \eta = \frac{3}{8}\sqrt{(2px)} = \frac{3}{8}y.$$

Making use of the values of ξ and η, it is easy to calculate by means of Guldin's theorem the volume of the body of revolution generated by the rotation of the considered figure around the coordinate axes or around a finite ordinate. For instance, in the last case the required volume is $V = 8\pi x^2 y/15$, since the distance of the centre of mass from the axis of revolution is $2x/5$.

(2) To calculate the centre of mass of the figure bounded by a branch of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$ and the x-axis.

Making use of Sec. 196, *(4)* and Sec. 198, *(2)*, we easily obtain from Guldin's theorem $\eta = 5a/6$; by symmetry $\xi = \pi a$.

208. Mechanical work. Suppose that a point M moves along a straight line (for simplicity we confine ourselves to this case only) and for a displacement s along this line there acts a constant force

F on the point in the direction of the line. It is known from the fundamentals of mechanics that the work W done by the force is given by the product $F \cdot s$.

However, in many cases the magnitude of the force is not constant but changes continuously with the position and then we again have to use a definite integral to express the work.

We take the distance s covered by the point as the independent variable; then, let the initial position A of the point M correspond to the value $s = s_0$, and the final position B to the value $s = S$ (Fig. 93). To every value of s in the interval $[s_0, S]$ there corresponds

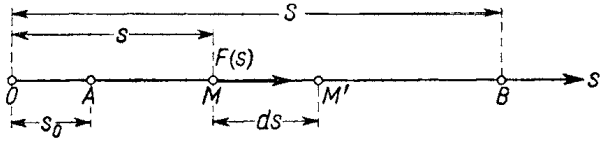

FIG. 93.

a definite position of the moving point, and also a definite value of the force F which can therefore be regarded as a function of s. Considering the point M in one of its positions defined by the value s of the traversed distance, we find an approximate expression for the element of work corresponding to an increment ds of the distance from s to $s + ds$ (i.e. corresponding to the movement of point M to the nearby position M') (see Fig. 93). At M the point is subjected to the force F; since the change in this quantity when passing from M to M' is small for a small ds, we shall disregard this change and suppose the force F to be approximately constant. Then we obtain for the element of work over the displacement ds the expression

$$dW = F \cdot ds.$$

Thus, the total work is given by the integral

$$W = \int_{s_0}^{S} F\,ds. \tag{10}$$

Example. Let us, for instance, apply formula (10) to calculate the work of compression (or elongation) of a spring with one end fixed (Fig. 94); this case arises, for instance, in designing buffers of railway carriages.

It is known that the elongation s of the spring (provided it is not overloaded) produces a tension p the magnitude of which is proportional to the elongation, i.e. $p = cs$ where c is a constant depending on the elastic properties of the spring (the "rigidity" of the spring). The force elongating the spring should overcome this tension. Taking into account only the part of the force which does this, we

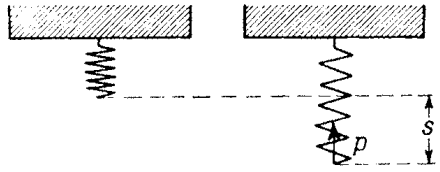

FIG. 94.

find that the corresponding work in increasing the elongation from $s_0 = 0$ to S is given by the integral

$$W = \int_0^S p\,ds = c\int_0^S s\,ds = \frac{cS^2}{2}.$$

Denoting by P the greatest value of the tension, or the force overcoming it, corresponding to the elongation of the spring (and equal to cS) we can express the work in the form

$$W = \tfrac{1}{2}PS.$$

If the force P were suddenly applied to the free end of the spring (for instance, by the suspension of a weight) over a displacement S, twice as much work PS is done. We observe that only half of it is used in elongating the spring; the other half provides the spring and the weight with kinetic energy.

CHAPTER 13

SOME GEOMETRIC APPLICATIONS OF THE DIFFERENTIAL CALCULUS

§ 1. The tangent and the tangent plane

209. Analytic representation of plane curves. In this chapter we shall consider some examples of a few applications of the differential calculus to geometry, mainly in the plane. These applications are investigated in detail in differential geometry, which is an independent subject.

We first recall various methods of analytic representation of curves on the plane (known to the reader from analytic geometry), assuming that a rectangular system of coordinates has been selected†.

(1) We have already examined equations of the form

$$y = f(x) \quad [\text{or } x = g(y)] \tag{1}$$

and we investigated the corresponding curve. This way of prescribing the curve, when one of the coordinates of its point is directly represented as a single-valued function of the other coordinate, is called "an explicit representation of the curve".

As an example, we mention the parabola $y = ax^2$.

(2) In analytic geometry the curve is usually given by an equation solved neither for x nor for y:

$$F(x, y) = 0; \tag{2}$$

this is called "an implicit equation of the curve".

---

† It is assumed that all functions to be considered in the present chapter are, as a rule, continuous and have continuous first derivatives with respect to their arguments; if necessary, we shall require the existence and continuity of higher derivatives.

Example. The ellipse $x^2/a^2 + y^2/b^2 = 1$. Sometimes we can express one variable in terms of the other from the equation (2), for instance y by x, and to represent the curve (or a part of it) by an explicit equation (1). Thus in the case of the ellipse

$$y = \pm \frac{b}{a} \sqrt{(a^2 - x^2)} \quad (\text{for } -a \leqslant x \leqslant a).$$

In other cases although the dependence of, say, y on x is described by an equation (2) and, under certain conditions,† there exists a single-valued function (1) which satisfies equation (2), and even this "implicit" function is continuous and has a continuous derivative, we cannot write down an explicit expression for it. Thus, for instance, in the case of the *trisectrix* we have $x^3 + y^3 - 3axy = 0$ (Fig. 95).

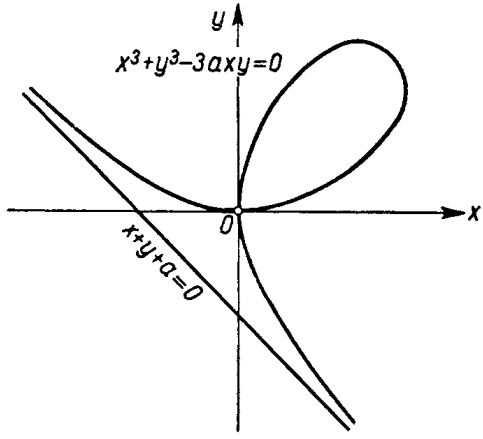

FIG. 95.

(3) Finally, we remarked above that an equation of the form

$$x = \varphi(t), \quad y = \psi(t), \tag{3}$$

establishing the dependence of the current coordinates of a point on a parameter t, also determines a curve on the plane. These equations are called parametric; they provide us with a parametric representation of the curve.

For instance, for the ellipse we have the parametric representation

$$x = a\cos t, \quad y = b\sin t.$$

When the parameter t varies (its geometric meaning is clear from Fig. 96) from zero to 2π the ellipse is described counter-clockwise, beginning from the end $A(a, 0)$ of the major axis.

† See Chapter 19 of the second volume.

As a second example consider the familiar cycloid

$$x = a(t - \sin t), \quad y = a(1 - \cos t),$$

which represents the locus of the point of a circle which rolls upon a straight line (Fig. 97). As the parameter, we have here the angle $t = \sphericalangle NDM$ between the movable radius DM and its initial position OA. When t varies from zero to 2π, the point describes a branch of the cycloid as shown in the figure. The whole curve corresponding to the variation of t from $-\infty$ to $+\infty$ consists of an infinite set of these arcs.

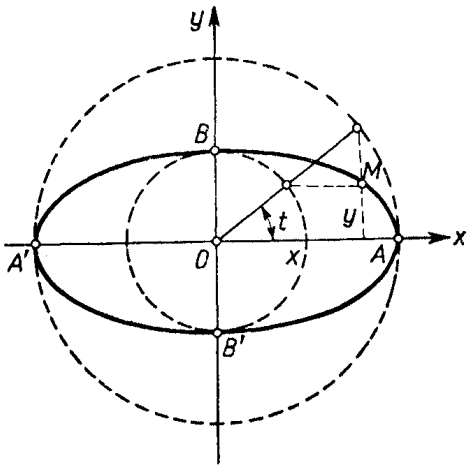

FIG. 96.

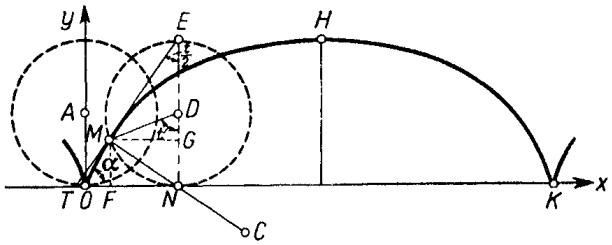

FIG. 97.

210. Tangent to a plane curve. The concept of a tangent has frequently been encountered [see, for instance, Sec. 77]. The curve represented by the explicit equation

$$y = f(x)$$

has at all points (x, y) a tangent the gradient of which, $\tan\alpha$, is given by the formula

$$\tan\alpha = y'_x = f'(x).$$

Thus, the equation of the tangent has the form

$$Y - y = y'_x(X - x). \tag{4}$$

Here (and henceforth) X, Y denote the current coordinates and x, y the coordinates of the point of contact.

It is easy to derive the equation of the *normal*, i.e. the straight line passing through the point of contact and perpendicular to the tangent:

$$Y - y = -\frac{1}{y'_x}(X - x)$$

or

$$X - x + y'_x(Y - y) = 0. \tag{5}$$

In connexion with the tangent and the normal we can examine certain segments—*TM* and *MN*—and their projections *TP* and *PN* on the x-axis (Fig. 98). The latter are called the *subtangent* and *subnormal*,

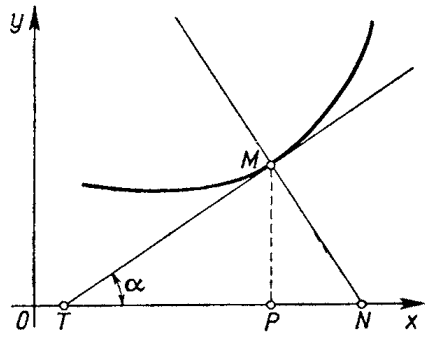

FIG. 98.

respectively and are denoted by sbt and sbn. Setting $Y = 0$ in equations (4) and (5) it is easy to see that

$$\text{sbt} = TP = \frac{y}{y'_x}, \quad \text{sbn} = PN = yy'_x. \tag{6}$$

Example (1) For instance, for the parabola $y = ax^2$ we have

$$\text{sbt} = \frac{y}{y'_x} = \frac{ax^2}{2ax} = \frac{x}{2},$$

a result which we already know (see the footnote on p. 144).

We now proceed to consider an implicit prescription of the curve by relation (2). Assuming that this equation is equivalent to an equation of the type (1) near the considered point†, then the curve will have a tangent (4) at this point. In Sec. 141, *(4)* we studied the representation of the derivative y'_x of an "implicit" function which we did not actually know, in terms of the known derivatives F'_x and F'_y; we have

$$y'_x = -\frac{F'_x(x, y)}{F'_y(x, y)},$$

assuming that $F'_y \neq 0$. (We note, incidentally, that this is precisely the condition under which equation (2) is equivalent to an equation of the form (1) in the neighbourhood of the considered point.) Substituting the above expression for y'_x in the equation of the tangent, after simple transformation we find

$$F'_x(x, y)(X-x) + F'_y(x, y)(Y-y) = 0. \tag{7}$$

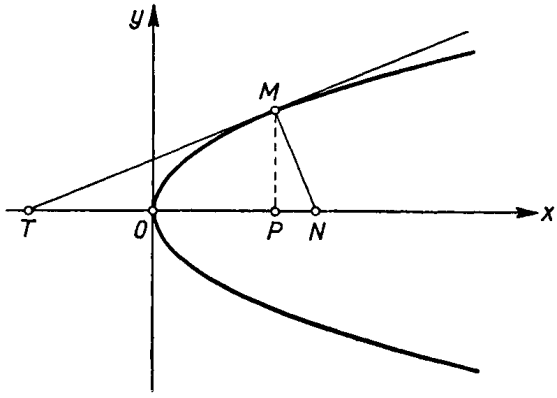

FIG. 99.

By the symmetry of the equation with respect to x and y it is evident that the same equation is obtained for the tangent if x and y are exchanged, now assuming that $F'_x \neq 0$. Only if both derivatives F'_x, F'_y simultaneously vanish at the considered point, relation (7) is converted into an identity and is no longer an equation of a definite straight line. In this case the point (x, y) is said to be a singular

† See the footnote on p. 424.

point of the curve; at a singular point a curve can, in fact, have no definite tangent.

Examples. *(2)* The parabola $y^2 = 2px$ (Fig. 99). Differentiating this relation and regarding y as a function of x we obtain $yy'_x = p$. Thus (see (6)) the subnormal of the parabola is a constant quantity. This indicates a simple method of constructing the normal and hence the tangent to the parabola.

Incidentally, in this case the subtangent is also expressed simply by dividing the equation of the parabola by the above relation; then we have

$$\frac{y}{y'_x} = 2x \quad \text{or} \quad \text{sbt} = 2x.$$

(3) The ellipse $x^2/a^2 + y^2/b^2 = 1$ (Fig. 100).

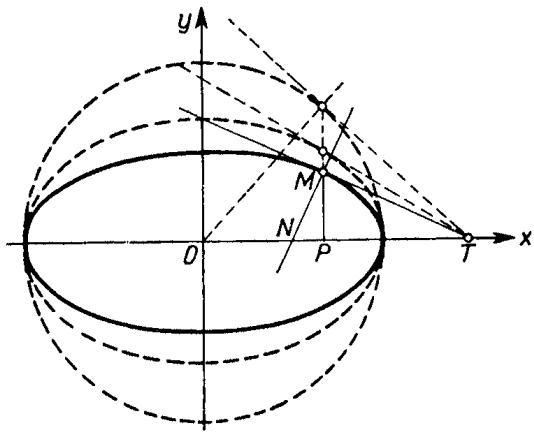

FIG. 100.

By formula (7) we have the equation of the tangent

$$\frac{x}{a^2}(X-x) + \frac{y}{b^2}(Y-y) = 0.$$

Taking into account the equation of the ellipse itself we can simplify the last relation:

$$\frac{xX}{a^2} + \frac{yY}{b^2} = 1.$$

Setting $Y = 0$ we obtain $X = a^2/x$. Thus, the point T of the intersection of the tangent with the x-axis is independent of y and b. Tangents to various ellipses corresponding to various values of b at points with abscissae x all pass through the same point T on the x-axis. Since for $b = a$ we have a circle for which the construction of the tangent is simple, point T is at once determined, which in turn

leads to a simple method of constructing the tangent to the ellipse; the method is clear from the figure.

(4) For the trisectrix $x^3+y^3-3axy=0$ both partial derivatives of the left-hand side of the equation

$$3(x^2-ay) \quad \text{and} \quad 3(y^2-ax)$$

vanish simultaneously at the origin; it is clear from the figure that there is in fact no definite tangent at this singular point of the curve.

Finally, let us examine a curve described by the parametric equations (3). If at a selected point the derivative $x'_t=\varphi'(t)$ is non-zero and is, say, positive, it is positive near this point; consequently the function $x=\varphi(t)$ increases monotonically [Sec. 111] and t is also an increasing function of x, i.e. $t=t(x)$ [Sec. 71], the derivative of which is $t'_x=1/x'_t$ [Sec. 80]. Substituting this function of x for t in the equation $y=\psi(t)$, we find that on a segment of the curve y is a function of x,

$$y=\psi(t(x))=f(x),$$

which also has a derivative. Therefore, a segment of the curve in a neighbourhood of the considered point can be expressed by an explicit equation: in this case the curve has a tangent at this point.

The gradient of the tangent can be expressed as follows:

$$\tan\alpha=y'_x=\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{y'_t}{x'_t}. \tag{8}$$

Substituting this expression in (4) we easily transform it to the form of the ratio

$$\frac{X-x}{x'_t}=\frac{Y-y}{y'_t}. \tag{9}$$

Incidentally, both denominators are frequently multiplied by dt and the equation of the tangent is written in the form

$$\frac{X-x}{dx}=\frac{Y-y}{dy}. \tag{10}$$

If we assumed that at the selected point the derivative $y'_t=\psi'(t)$ does not vanish, then, exchanging x and y, we would have arrived at the same equation of the tangent. Only when both derivatives

x_t' and y_t' simultaneously vanish at the considered point is our reasoning invalid. This point is also called a singular point of the curve; there can be no tangent at this point. Incidentally, relations (9) and (10) are then meaningless, since both denominators vanish.

(5) As an example consider the problem of constructing the tangent to the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$ (Fig. 97). In this case we have

$$x_t' = a(1 - \cos t), \quad y_t' = a \sin t,$$

and the singular points correspond to $t = 2k\pi$ ($k = 0, \pm 1, \pm 2, \ldots$). Excluding these points we have, by (8),

$$\tan \alpha = \frac{\sin t}{1 - \cos t} = \cot \frac{t}{2} = \tan\left(\frac{\pi}{2} - t\right),$$

and we may set $\alpha = (\pi/2) - t$.

Recall that (Fig. 97) $t = \sphericalangle MDN$ and hence $\sphericalangle MEN = t/2$. If the straight line EM be continued to intersect the x-axis at T, then $\sphericalangle ETx = (\pi/2) - t = \alpha$. Consequently the straight line ME connecting a point of the cycloid with the highest point of the rolling circle in the current position is the tangent. It is therefore clear that the straight line MN is the normal.

Subsequently we shall employ the expression for the segment n of the normal to the intersection with the x-axis, which can easily be deduced from the right-angled triangle MEN. Thus we have

$$n = MN = 2a \sin \frac{t}{2}.$$

Now tangents exist even at the singular points—they are parallel to the y-axis; however, the location of the curve itself with respect to the tangents at these points is unusual: cusps ("recurrent points") occur there.

211. Positive direction of the tangent. So far, we have been determining the position of the tangent to a curve by its gradient $\tan \alpha$ without distinguishing the two opposite directions on the tangent itself; $\tan \alpha$ is the same for both cases. However, in some investigations it is necessary to fix one of these directions.

Consider a curve given by the parametric equation (3) and an "ordinary" point on it (i.e. a non-singular point). We know from Sec. 202 that the derivatives

$$x_s' = \frac{dx}{ds}, \quad y_s' = \frac{dy}{ds}$$

exist at this point, and

$$\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2=1; \tag{11}$$

this relation can easily be derived from the basic relation

$$dx^2+dy^2=ds^2$$

[Sec. 202, (6)] by dividing the latter by ds^2.

Before proceeding to the essence of the problem indicated in the title of the section, we establish an auxiliary proposition which will later be useful.

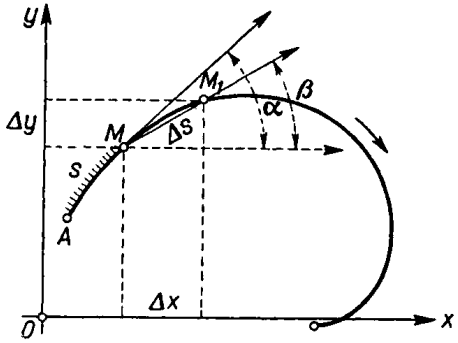

FIG. 101.

LEMMA. *Suppose that M is an ordinary point of the curve (Fig. 101). Denoting by M_1 a variable point of the considered curve, when M_1 tends to M the ratio of the length of the chord MM_1 to the length of the arc $\overset{\frown}{MM_1}$ tends to unity:*

$$\lim_{M_1\to M}\frac{MM_1}{\overset{\frown}{MM_1}}=1. \tag{12}$$

Take the arc as the parameter and suppose that the point M corresponds to a value s of the arc, and the point M_1 to a value $s+\Delta s$. Let their coordinates be x, y and $x+\Delta x, y+\Delta y$, respectively. Then $\overset{\frown}{MM_1}=|\Delta s|$ and $MM_1=\sqrt{(\Delta x^2+\Delta y^2)}$. Hence

$$\frac{MM_1}{\overset{\frown}{MM_1}}=\frac{\sqrt{(\Delta x^2+\Delta y^2)}}{|\Delta s|}=\sqrt{\left[\left(\frac{\Delta x}{\Delta s}\right)^2+\left(\frac{\Delta y}{\Delta s}\right)^2\right]}.$$

Letting $\Delta s\to 0$ and using (11), we arrive at the required result.

Thus, under the indicated conditions, an arc and the corresponding infinitesimal chord are equivalent.

Now suppose that we have selected the initial point on the considered curve and also a definite direction of measuring the arc; again take the arc as the parameter determining the position of the point on the curve.

Suppose that the considered point M corresponds to arc s. If we let s have a positive increment Δs, then the arc $s+\Delta s$ determines a new point M_1 lying in the direction of increasing arcs from M. The secant is directed from M to M_1, and the angle between this direction of the secant and the positive direction of the x-axis is denoted by β. Projecting the segment MM_1 on the coordinate axes (Fig. 101), according to a familiar theorem from the theory of projections we obtain

$$\text{pr.}_x MM_1 = \Delta x = MM_1 \cos\beta, \quad \text{pr.}_y MM_1 = \Delta y = MM_1 \sin\beta,$$

whence

$$\cos\beta = \frac{\Delta x}{MM_1}, \quad \sin\beta = \frac{\Delta y}{MM_1}.$$

Since $\overset{\frown}{MM_1} = \Delta s$ these relations may be rewritten as follows:

$$\cos\beta = \frac{\Delta x}{\Delta s}\,\frac{\overset{\frown}{MM_1}}{MM_1}, \quad \sin\beta = \frac{\Delta y}{\Delta s}\,\frac{\overset{\frown}{MM_1}}{MM_1}. \tag{13}$$

We regard as positive the direction of the tangent which coincides with increasing arcs; strictly speaking it is defined as the limiting position as $\Delta s \to 0$, of the ray MM_1 constructed as above. If the angle between the positive direction of the tangent and the positive direction of the x-axis be denoted by α, we obtain from (13) in the limit in view of (12)

$$\cos\alpha = \frac{dx}{ds}, \quad \sin\alpha = \frac{dy}{ds}. \tag{14}$$

These formulae determine the angle α to within $2k\pi$ (k is an integer) and consequently they in fact fix one of the two possible directions of the tangent, namely the positive direction.

212. The case of a spatial curve. This problem will be only briefly examined, since it is completely analogous with the case of a plane curve.

As in the case of a plane curve, the coordinates of the variable point of the spatial curve can be given as functions of an auxiliary variable—the parameter t,

$$x = \varphi(t), \quad y = \psi(t), \quad z = \chi(t), \tag{15}$$

in such a way that when the parameter t varies, the point whose coordinates are given by this relation describes the considered curve.

In the case of a spatial curve (15), the definition of the tangent is the same as for the plane curve. We exclude from our considerations singular points of the curve defined as those points at which the derivatives x'_t, y'_t, z'_t vanish simultaneously, and consider an ordinary point $M(x, y, z)$ of the curve determined by the value t of the parameter. Let t have an increment Δt; then there corresponds to the new value $t + \Delta t$ of the parameter another point $M_1(x + \Delta x, y + \Delta y, z + \Delta z)$. The equations of the secant MM' have the form

$$\frac{X - x}{\Delta x} = \frac{Y - y}{\Delta y} = \frac{Z - z}{\Delta z},$$

where X, Y, Z are the current coordinates. The geometric meaning of these equations is unaltered if all the denominators are divided by Δt:

$$\frac{X - x}{\frac{\Delta x}{\Delta t}} = \frac{Y - y}{\frac{\Delta y}{\Delta t}} = \frac{Z - z}{\frac{\Delta z}{\Delta t}}.$$

If these equations have a definite meaning in the limit, this establishes the existence of the limiting position of the secant, i.e. that of the tangent†. But, in the limit, we have

$$\frac{X - x}{x'_t} = \frac{Y - y}{y'_t} = \frac{Z - z}{z'_t}, \tag{16}$$

and these equations in fact express a straight line, since not all the denominators vanish. Thus, at every ordinary point of the curve the tangent exists and is expressed by these equations. For a singular point the problem of the tangent remains unsolved.

† We have passed to the limit $\Delta t \to 0$, but it can be proved that this is equivalent to the "more geometric" relation $MM_1 \to 0$.

Sometimes it is convenient to write equations (16) in the form

$$\frac{X-x}{dx}=\frac{Y-y}{dy}=\frac{Z-z}{dz}, \tag{17}$$

which is derived from (16) by multiplication of all the denominators by dt.

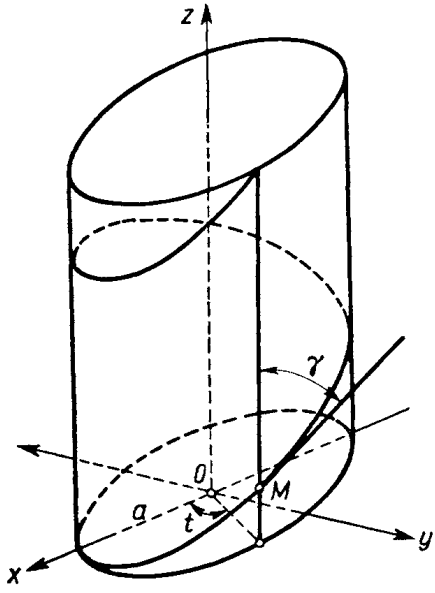

FIG. 102.

Denoting by α, β, γ the angles between the tangent and the coordinate axes, the direction cosines $\cos\alpha$, $\cos\beta$, $\cos\gamma$ take the form

$$\cos\alpha=\frac{x'_t}{\pm\sqrt{(x'^2_t+y'^2_t+z'^2_t)}}$$

$$\cos\beta=\frac{y'_t}{\pm\sqrt{(x'^2_t+y'^2_t+z'^2_t)}},$$

$$\cos\gamma=\frac{z'_t}{\pm\sqrt{(x'^2_t+y'^2_t+z'^2_t)}}.$$

The choice of a definite sign of the root corresponds to the choice of a definite direction of the tangent.

As an example consider the helix (Fig. 102)

$$x=a\cos t, \quad y=a\sin t, \quad z=ct.$$

Here

$$x'_t = -a\sin t, \quad y'_t = a\cos t, \quad z'_t = c,$$

and the equations of the tangent have the form

$$\frac{X-x}{-a\sin t} = \frac{Y-y}{a\cos t} = \frac{Z-z}{c}.$$

The direction cosines of the tangent are

$$\cos\alpha = -\frac{a\sin t}{\sqrt{(a^2+c^2)}}, \quad \cos\beta = \frac{a\cos t}{\sqrt{(a^2+c^2)}}, \quad \cos\gamma = \frac{c}{\sqrt{(a^2+c^2)}}.$$

Note that $\cos\gamma = \text{const}$, and consequently $\gamma = \text{const}$. If we regard the helix as wound around a right circular cylinder we see that it intersects the generators of the cylinder at a constant angle.

As in the case of a plane curve, we may select the arc s measured from an arbitrary point (in a definite direction) for the parameter determining the position of a point of a spatial curve; for the positive direction we select the one corresponding to increasing arcs. If the considered point is ordinary, the direction cosines of the tangent with positive direction have the form

$$\cos\alpha = \frac{dx}{ds}, \quad \cos\beta = \frac{dy}{ds}, \quad \cos\gamma = \frac{dz}{ds} \tag{18}$$

[see Sec. 211].

213. The tangent plane to a surface. We have examined already [Sec. 124] a surface given by the equation

$$z = f(x, y), \tag{19}$$

This is an explicit equation of the surface†.

In analytic geometry the surface is more often given by the implicit equation

$$F(x, y, z) = 0, \tag{20}$$

which is not solved with respect to any variable.

Examples.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1 = 0 \text{ (an ellipsoid)},$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2} = 0 \quad \text{(a cone of the second order)}.$$

† Of course, the exceptional role of z is incidental; description of the surface in the form $x = g(y, z)$ or $y = h(x, z)$ is also explicit.

As in the case of an implicit description of a plane curve, under certain conditions† here, too, relation (20) turns out to be equivalent to an equation of the form (19), determining one coordinate as a function of the other two (with continuous partial derivatives), although we may not know the explicit expression for this function.

Suppose that $M(x, y, z)$ is a point of the surface (20). Draw an arbitrary curve on the surface through M and, at M, construct a tangent to this curve; there exists an infinite set of such curves (and tangents to them).

If all the tangents at the point M to the various curves drawn through this point on the surface lie in one plane, the plane itself is called the *tangent plane* to the surface at the point M; the point M is then said to be the point of contact.

A curve drawn on the surface (20) can, in general, be represented analytically by equations of the form (15). Since, according to the assumption, the curve lies on the surface (all its points lie on the surface), substituting in (20) the functions φ, ψ, χ instead of x, y, z respectively, the equation is converted into an identity in the parameter t. Differentiating with respect to t we obtain (making use of the invariance of the form of the first differential, Sec. 143)

$$F'_x\,dx + F'_y\,dy + F'_z\,dz = 0, \tag{21}$$

where for the arguments of the functions F'_x, F'_y, F'_z we may, in particular, take the coordinates x, y, z of the point of contact M, and dx, dy, dz are the differentials of the functions (15) for the corresponding value of t. On the other hand, the tangent to the considered curve at the point $M(x, y, z)$ is given by equations (17) where X, Y, Z are the current coordinates and dx, dy, dz denote the same quantities as above. Substituting in (21) the proportional (by (17)) differences $X-x, Y-y, Z-z$ for dx, dy, dz we finally obtain the relation

$$F'_x(X-x) + F'_y(Y-y) + F'_z(Z-z) = 0, \tag{22}$$

which is therefore satisfied at all points of an arbitrary tangent (mentioned in the definition). If at least one of the derivatives F'_x,

† See the footnote on p. 424.

F'_y, F'_z does not vanish at the point M, relation (22) represents an equation of the tangent plane.

In the exceptional case, when at the considered point we have simultaneously

$$F'_x = F'_y = F'_z = 0,$$

(such a point is called *singular*), relation (22) becomes an identity and the tangent plane may not exist.

Examples. *(1)* The ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

The tangent plane is obtained from formula (22), and the equation of the ellipse itself, in the form

$$\frac{xX}{a^2} + \frac{yY}{b^2} + \frac{zZ}{c^2} = 1.$$

(2) The cone of first order

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$$

The tangent plane

$$\frac{xX}{a^2} + \frac{yY}{b^2} - \frac{zZ}{c^2} = 0.$$

At the vertex $(0, 0, 0)$ of the cone, which is its only singular point, the equation is meaningless and there is no tangent plane.

The direction cosines of the normal to the surface (i.e. the perpendicular to the tangent plane at the point of contact) are obviously

$$\cos\lambda = \frac{F'_x}{\pm\sqrt{(F'^2_x + F'^2_y + F'^2_z)}}, \quad \cos\mu = \frac{F'_y}{\pm\sqrt{(F'^2_x + F'^2_y + F'^2_z)}},$$

$$\cos\nu = \frac{F'_z}{\pm\sqrt{(F'^2_x + F'^2_y + F'^2_z)}}.$$

The explicit equation (19) in the form

$$z - f(x, y) = 0$$

can be regarded as a particular case of (20). Introducing the ordinary notation

$$\frac{\partial f}{\partial x} = p, \quad \frac{\partial f}{\partial y} = q,$$

the equation of the tangent plane (22) for this case takes the form

$$Z - z = p(X - x) + q(Y - y), \tag{23}$$

and the direction cosines of the normal are

$$\left.\begin{aligned} \cos\lambda &= \frac{-p}{\pm\sqrt{(1+p^2+q^2)}}, \quad \cos\mu = \frac{-q}{\pm\sqrt{(1+p^2+q^2)}}, \\ \cos\nu &= \frac{1}{\pm\sqrt{(1+p^2+q^2)}}. \end{aligned}\right\} \tag{24}$$

§ 2. Curvature of a plane curve

214. The direction of concavity, points of inflection. We consider a plane curve given, say, by an explicit equation $y=f(x)$, and a point $M(x_0, f(x_0))$ on the curve.

We say that the curve is concave in a definite direction from the tangent, at the point M, if in a sufficiently small neighbourhood of the point M all points of the curve lie exactly in this direction from the

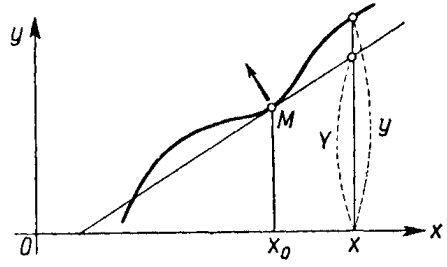

FIG. 103.

tangent (Fig. 103). A point is called a point of inflection if—again in a sufficiently small neighbourhood of it—the points of the curve with the abscissae $x<x_0$ lie on one side of the tangent, while points with the abscissae $x>x_0$ lie on the other side, i.e. if at the point M the curve passes from one side of the tangent to the other side or, briefly, if it intersects the tangent (Fig. 104).

Since the equation of the tangent at the point M is

$$Y=f(x_0)+f'(x_0)(x-x_0)^{\dagger},$$

† We have changed the notation here from that used in Sec. 210 (see (4)) but the current ordinate of the point of contact of the tangent has as before been denoted by Y in order to distinguish it from the ordinate $y=f(x)$ of the point of the curve with the same abscissa x.

to determine the direction of concavity or the presence of a point of inflection we have to investigate the sign of the difference

$$y - Y = f(x) - f(x_0) - f'(x_0)(x - x_0)$$

in the neighbourhood of the point x_0. We assume the existence in this neighbourhood of the continuous second derivative $f''(x)$.

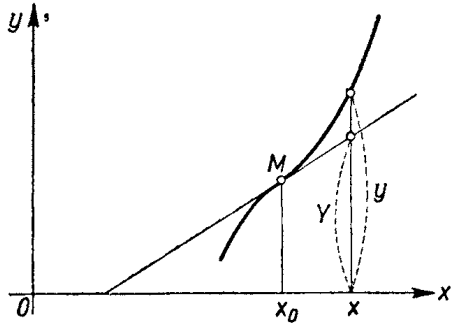

FIG. 104.

First suppose that $f''(x_0) \neq 0$. Making use of the Taylor formula with the remainder term in Peano's form [Sec. 107, (17)] for $n = 2$ we obtain

$$y - Y = \frac{f''(x_0) + \alpha}{2!}(x - x_0)^2,$$

where $\alpha \to 0$ as $x \to x_0$. For values of x sufficiently close to x_0 this difference has the sign of the number $f''(x_0)$ and consequently, at the point M, the curve is concave upwards if $f''(x_0) > 0$ and concave downwards if $f''(x_0) < 0$.

If $f''(x_0) = 0$ the term $\alpha/2$ remains on the left, which does not tell us anything about the sign of the difference $y - Y$. In this case we use Lagrange's form of the remainder term [Sec. 106, (12)] also for $n = 2$:

$$y - Y = \frac{f''(c)}{2!}(x - x_0)^2$$

Here either $x < c < x_0$ or $x_0 < c < x$. If near the value x_0 the second derivative $f''(x)$ has the same sign for x on both sides of x_0, then the difference also has this same sign on both sides of x_0, and M is a point of concavity upwards or downwards, respectively, as the sign is positive or negative.

Conversely, if $f''(x)$ changes its sign when passing through the point x_0, then the difference $y - Y$ also changes sign and we have a point of inflection at M. In this case the point of inflection M, provided we confine ourselves to a sufficiently small neighbourhood of it, separates the points at which the concavity is directed upwards from those points at which the concavity is downwards†.

As an example we consider the sinusoid $y = \sin x$; here $y'' = -\sin x = -y$. Consequently, in the intervals where $\sin x$ has a positive (negative) sign the concavity of the sinusoid is downwards (upwards). For the values of the form $x = k\pi$ (k is an integer) y'' vanishes while changing sign; here we therefore have points of inflection of the sinusoid. On the other hand, for the function $y = x^4$ we have $y'' = 12x^2$ and although at $x = 0$ the second derivative vanishes, for all other values of x it has a positive sign and the concavity of the curve is upwards everywhere.

Assuming the existence of the second derivative, the condition $y'' = 0$ is necessary but not sufficient for the presence of a point of inflection.

The analogy with the theory of extrema is readily observed [Sec. 112 *et seq.*].

Finally, observe that instead of investigating the sign of the second derivative $f''(x)$ near the point x_0, we can alternatively investigate the successive derivatives $f'''(x_0), f^{(4)}(x_0), \ldots$ at the point x_0 itself. Since the relevant reasoning is identical with that of Sec. 117, we leave it to the reader.

Remark. Investigation of the presence of points of inflection on the curve enables one to specify the graph of the function more precisely than was done in Sec. 115.

215. The concept of curvature. We consider an arc of a curve without multiple or singular points, given by the parametric equations

$$x = \varphi(t), \quad y = \psi(t). \tag{1}$$

If we draw the tangent (say in the positive direction), at all points of the curve, on account of the "bend" of the curve the tangent rotates as the point of contact is displaced; this is an essential difference between a curve and a straight line for which the tangent (coinciding with the line) has one direction for all points.

† Sometimes this property is used to define a point of inflection. This definition is equivalent to that given above.

An important property describing the behaviour of the curve is the "degree of bend" or the "curvature" at various points; this curvature can be expressed by a number.

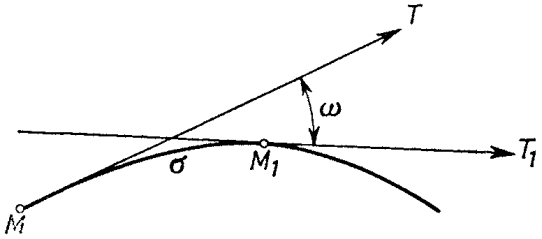

FIG. 105.

Let $\widetilde{MM_1}$ (Fig. 105) be an arc of a curve; consider the tangents MT and M_1T_1 drawn (in the positive direction) at the ends of the arc. It is natural to describe the curvature of the curve by the angle of rotation of the tangent per unit length of the arc, i.e. by the ratio ω/σ where the angle ω is measured in radians and the length σ in some selected units of length. This ratio is called "the mean curvature of an arc of a curve".

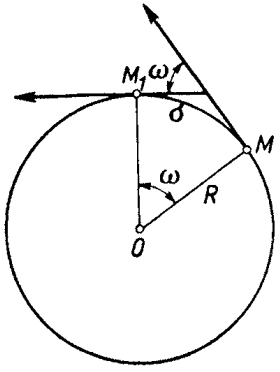

FIG. 106.

On various segments of the curve its mean curvature is in general different. There exists (as a matter of fact, it is unique) a curve for which the mean curvature is everywhere the same; this is the circle†. In fact, we have in this case (Fig. 106)

$$\frac{\omega}{\sigma} = \frac{\omega}{R\omega} = \frac{1}{R}$$

for any arc.

† That is, besides the straight line the curvature of which is everywhere zero.

The concept of the mean curvature of an arc $\overset{\frown}{MM_1}$ leads to the concept of the *curvature at a point.*

By the *curvature* at a point M of an arc we understand the limit to which the mean curvature of the arc $\overset{\frown}{MM_1}$ tends when the point M_1 approaches M along the curve.

Denoting the curvature at a point by the symbol k we have

$$k = \lim_{\sigma \to 0} \frac{\omega}{\sigma}.$$

It is evident that for the circle $k = 1/R$, i.e. the curvature of the circle is a quantity inversely proportional to the radius of the circle.

Remark. The concepts of mean curvature and curvature at a point are entirely analogous to the concepts of mean velocity and velocity at a given instant of time for a moving point. We may say that the mean curvature describes the mean velocity of variation of the direction of the tangent on an arc, and the curvature at a point describes the actual velocity of variation of this direction at the considered point.

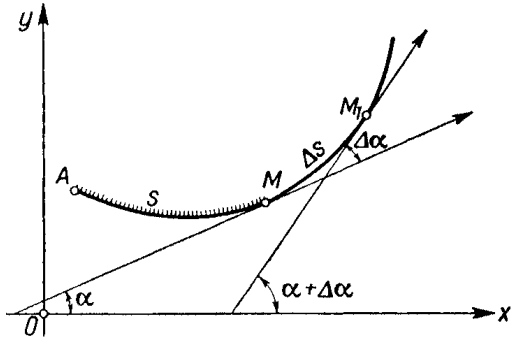

FIG. 107.

We now proceed to derive for the curvature an analytic expression which will enable us to calculate it from the parametric equations of the curve.

We first take the parameter as the arc (length). Take on the curve an ordinary point M and suppose that it corresponds to the value s of the arc. Letting s have an arbitrary increment Δs we obtain another point $M_1(s+\Delta s)$ (Fig. 107). The increment $\Delta\alpha$ of

the angle of inclination of the tangent when passing from M to M_1 gives the angle ω between the two tangents, so $\omega = \Delta\alpha$. Since $\sigma = \Delta s$, the mean curvature is equal to $\Delta\alpha/\Delta s$.

When $\widetilde{MM_1} = \Delta s$ tends to zero, we obtain the expression

$$k = \frac{d\alpha}{ds} \tag{2}$$

for the curvature of the curve at the point M.

It is important to note that this formula is valid only to within the sign, since by our definition the curvature is a non-negative number, while a negative number may occur on the right-hand side. The reason is that since both $\Delta\alpha$ and Δs may be negative, strictly speaking we should write $\omega = |\Delta\alpha|$, $\sigma = |\Delta s|$ and finally

$$k = \left|\frac{d\alpha}{ds}\right|.$$

This remark should henceforth be borne in mind.

To express (2) in a more convenient form for calculations (and at the same time to establish the very existence of the curvature) we now assume that the functions φ and ψ appearing in the parametric equations of the curve (1) have continuous derivatives of the first two orders.

If the point $M(t)$ is ordinary, without loss of generality we may assume that $x'_t = \varphi'(t) \neq 0$.

We now write formula (2) in the form

$$k = \frac{d\alpha}{ds} = \frac{\alpha'_t}{s'_t}. \tag{3}$$

But $s'_t = \sqrt{(x'^2_t + y'^2_t)}$ [Sec. 202, (5)]; it therefore remains to find α'_t. Since [Sec. 211, (8)]

$$\tan\alpha = \frac{y'_t}{x'_t} \quad \text{and} \quad \alpha = \arctan\frac{y'_t}{x'_t},$$

we have

$$\alpha'_t = \frac{1}{1+\left(\frac{y'_t}{x'_t}\right)^2}\,\frac{x'_t y''_{t^2} - x''_{t^2} y'_t}{x'^2_t} = \frac{x'_t y''_{t^2} - x''_{t^2} y'_t}{x'^2_t + y'^2_t}. \tag{4}$$

Substituting into (3) the values of s'_t and α'_t we arrive at the final formula

$$k = \frac{x'_t y''_{t^2} - x''_{t^2} y'_t}{(x'^2_t + y'^2_t)^{3/2}}. \tag{5}$$

This formula is quite suitable for calculations, since all the derivatives appearing in it are easily calculated from the parametric equations of the curve.

If the curve is given by the explicit equation $y = f(x)$, the formula takes the form

$$k = \frac{y''_{x^2}}{(1 + y'^2_x)^{3/2}}. \tag{5a}$$

Finally, for the case of the polar equation of the curve $r = g(\theta)$, we may as usual pass to the parametric representation in rectangular coordinates, taking θ as the parameter. Then with the help of (5) we obtain

$$k = \frac{r^2 + 2r'^2_\theta - rr''_{\theta^2}}{(r^2 + r'^2_\theta)^{3/2}}. \tag{5b}$$

216. The circle of curvature and radius of curvature. In various investigations it is convenient to replace approximately the curve near the considered point by a circle of the same curvature as the curve at that point.

By the circle† of curvature of the curve at the considered point M we understand the circle which

(1) touches the curve at the point M,

(2) has the concavity directed in the same direction as the curve at the point M,

(3) has the same curvature as the curve at the point M (Fig. 108).

The centre C of the circle of curvature is simply called the centre of curvature and its radius the radius of curvature (of the curve at the considered point).

It follows from the definition of the circle of curvature that the centre of curvature is always located on the normal to the curve at the considered point, and on the side of the concavity. Denoting

† Here "circle" means, of course, "circumference".

the curvature of the curve at the considered point by k, bearing in mind [Sec. 215] that for the circle $k = 1/R$, we evidently have now for the radius of curvature

$$R = \frac{1}{k}.$$

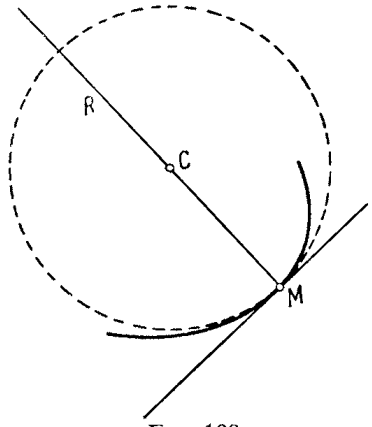

FIG. 108.

Making use of various expressions derived in the preceding section for the curvature, we can at once write down a number of formulae for the radius of curvature:

$$R = \frac{ds}{d\alpha}, \tag{6}$$

$$R = \frac{(x_t'^2 + y_t'^2)^{3/2}}{x_t' y_{t^2}'' - x_{t^2}'' y_t'}, \tag{7}$$

$$R = \frac{(1 + y_x'^2)^{3/2}}{y_{x^2}''}, \tag{7a}$$

$$R = \frac{(r^2 + r_\theta'^2)^{3/2}}{r^2 + 2r_\theta'^2 - r r_{\theta^2}''}, \tag{7b}$$

which will be applied when required.

The remark concerning the sign of the expression for the curvatuer [Sec. 215] also holds here.

Incidentally, instead of disregarding the sign we could interpret it geometrically, connecting it with the direction from the tangent

(positively directed—Sec. 211) of the radius of curvature along the normal at the point of contact. Thus, for the ordinary location of the coordinate axes the positive sign of the radius of curvature indicates that it is directed to the left from the tangent, while the negative sign indicates that it is directed to the right†. This can easily be verified in the case of explicit equation of the curve, since then (see (7a)) the sign of the radius of curvature is identical with the sign of y''_{x^2}, while the latter (as we know from Sec. 214) determines the direction of the concavity of the curve from the tangent (and therefore also the radius of curvature).

Examples. (1) To determine the radius of curvature of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$ (Fig. 97, p. 425).

Since [Sec. 210, *(5)*] $\alpha = (\pi/2) - t$, we have $ds = -dt/2$; on the other hand [Sec. 201, *(2)*] $\sqrt{(x'^2_t + y'^2_t)} = 2a \sin(t/2)$ i.e. $ds = 2a \sin(t/2)dt$. Now, to calculate R we use the basic formula (6)

$$R = \frac{ds}{d\alpha} = \frac{2a \sin\frac{t}{2}\, dt}{-\frac{1}{2}\, dt} = -4a \sin\frac{t}{2}.$$

Bearing in mind [Sec. 210, *(5)*] the expression for the segment of the normal to the intersection with the x-axis it turns out that

$$R = -2n.$$

This indicates a method of constructing the centre of curvature C; it is shown in the figure.

(2) To conclude, we briefly examine an applied problem in which we essentialy use the change of the curvature along the curve; the problem consists in investigating the so-called transition curves used in the division of railway curves.

It is known from mechanics that when a particle moves along a curve a centrifugal force arises, the magnitude of which is given by the formula

$$F = \frac{mv^2}{R},$$

where m is the mass of the particle, v its velocity and R the radius of curvature of the curve at the considered point.

If the straight part of a railway track were joined directly to the bend in the shape of an arc of a circle (Fig. 109*a*), in passing onto this bend the centrifugal force would be produced instantaneously, causing an impulse between the rolling stock and the rails. To eliminate this, the straight part of the track is connected

† We have to remember here that the positive direction of counting the arcs corresponds to the increasing of the parameter (t, x or θ).

to the circular part by means of a transition curve (Fig. 109*b*). Along the latter, the radius of curvature gradually decreases from infinity at the point of junction with the straight part to the magnitude of the radius of the circle at the junction with the circle, and accordingly the centrifugal force is created gradually.

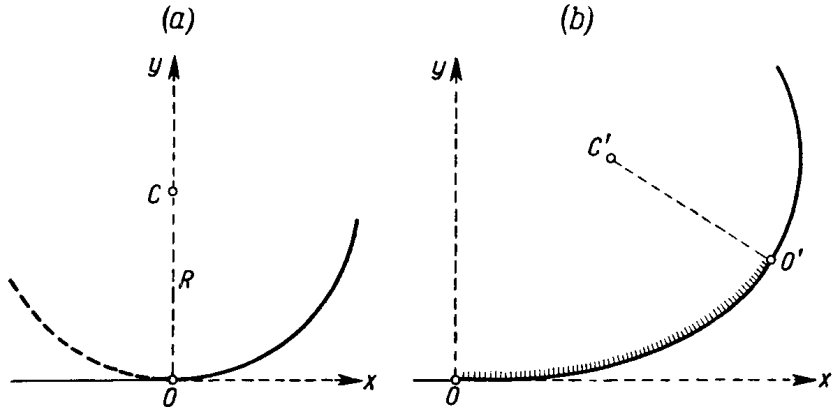

FIG. 109.

We may, for instance, use the cubic parabola $y = x^3/6q$ as the transition curve. It is evident that, in this case,

$$y' = \frac{x^2}{2q}, \quad y'' = \frac{x}{q}$$

Hence, for the radius of curvature we obtain

$$R = \frac{q}{x}\left(1 + \frac{x^4}{4q^2}\right)^{\frac{3}{2}}.$$

For $x = 0$ we have $y' = 0$ and $R = \infty$, and our curve is tangential to the x-axis at the origin and has zero curvature there.

CHAPTER 14

HISTORICAL SURVEY OF THE DEVELOPMENT OF THE FUNDAMENTAL CONCEPTS OF MATHEMATICAL ANALYSIS

§ 1. Early history of the differential and integral calculus

217. Seventeenth century and the analysis of infinitesimals. This was the time of the transition from the Middle Ages to modern times, the beginning of the flourishing of capitalism which in its struggle with the feudal system was a progressive force. Science received strong impulses from life itself. Navigation resulted in an increasing interest in astronomy and optics. Ship-building, the design of dams and canals, the construction of various machines and structures, the problems of ballistics and military requirements in general, furthered the development of mechanics. On the other hand, astronomy, optics, mechanics and engineering themselves demanded a decisive modernization of the mathematics of that time.

This modernization was affected by the introduction of the *variable quantity* which was rightly called by Engels "the turning point in mathematics" (see the quotation on p. 26). Only the mathematics of variable quantities could satisfy the demands of the developing mathematical sciences. New problems led to the introduction of fresh methods of investigation connected with the "infinitesimal quantities" (or the "infinitesimal methods"). Hence, at the end of the century mathematical analysis was converted into an independent science called the "analysis of infinitesimals"; this name has survived until now.

In the beginning "primitive methods" prevailed—establishing every single fact required a special procedure. However, in the course

of time the position changed. New methods were announced for solving problems of the same type, connections were found between problems of various types, gradually general concepts were elucidated and formed the basis of the solution; all these developments were brilliantly crowned by Newton and Leibniz in the creation of the differential and integral calculus.

The first section of the chapter is devoted to the survey of the accomplishments of at least two generations of mathematicians, who were preparing this discovery over a period of half a century.

218. The method of indivisibles. We begin with the early history of the integral calculus which, in fact, started in antiquity; in the calculation of areas and volumes and also the location of centres of mass of various figures the real predecessor of the mathematicians of the seventeenth century was Archimedes (third century B.C.).

In *Epistle of Archimedes to Eratosthenes*, which has been preserved, are stated all the preliminary results which Archimedes derived by a special method in which he formally used the theory of equilibrium of a lever, but the essence of which lay in the idea of constructing plane figures of lines and bodies of planes. The facts found by this "atomic" method were subsequently published, together with rigorous proofs based, in accordance with the custom of that time, on assuming the converse. However, the mathematicians of the seventeenth century did not know this *Epistle*, since for over two thousand years it was regarded as lost, its text being discovered entirely accidentally at the beginning of this century. Thus in the epoch we are considering now, information about the method used by Archimedes for discovering his results could only be obtained from other works of his: in the latter there was frequently no trace of the way in which the results were deduced. However, in some of his proofs, Archimedes employed the method of dividing a plane figure (or body) into elements but the number of them was finite and they were of finite thicknesses; in this connection he also examined the inscribed and escribed step figures (bodies) which constitute the geometric prototype of our integral sums.

The first attempt to rediscover the method of Archimedes and to extend its domain of application was carried out by a German astronomer and mathematician, Johann Kepler (1571–1630). He

published, in 1615, a book entitled *New Stereometry of Wine Barrels.* Although the work resulted from an incidental cause and it seems that the subject is purely practical, it contains a new method of approaching the problem of squaring and cubing: a plane figure is divided into an infinite number of infinitesimal elements, and then, out of these elements, deformed if required, a new figure is constructed, the area of which is known (and similarly for a volume).

It should be observed that the elements considered in Kepler's works are by no means devoid of thickness: he speaks of "most thin little circles", of "parts with extremely small width, as if linear", etc.

In this way Kepler first obtains direct results for a number of problems already known to Archimedes and subsequently, in the chapter called "An Appendix to Archimedean Works", he calculates the volumes of 87 new bodies of revolution.

A successor of Kepler's ideas and the originator of "the method of indivisibles" was an Italian scholar and priest, a pupil of Galileo, Bonaventura Cavaglieri (1598–1647) for whom the dissemination of this method became the purpose of his whole life. In 1635 his basic work was published, entitled *Geometry Exposed by a New Method of Indivisibles of a Continuous*; in 1647 he published the further work *Six Geometric Experiments.* Essentially these papers resurrect the "atomic" viewpoint of Archimedes.

"To determine the magnitude of plane figures", says Cavaglieri, "straight lines are applied, parallel to another straight line ..., which we imagine as infinite in number in these figures ..." (Fig. 110). Similarly, he tackles bodies, but instead of lines, planes are drawn. These lines (planes) are precisely the "indivisibles"; "their number is unbounded and they are devoid of any thickness" (in this respect Cavaglieri differs from Kepler). However, Cavaglieri was not bold enough to state that figures or bodies *consist* of these indivisibles, devoid of thickness. His fundamental proposition is formulated in a more cautious manner: "plane figures (or bodies) *are in the same relation* as all their indivisibles taken together". For instance, if the parallelogram $ABCD$ (Fig. 111) is divided by the diagonal into two triangles and straight lines parallel to the base CD are drawn, then "the ratio of all lines (OR) of the parallelogram" to

"all lines (QR) of the triangle" is 2 : 1, since this is the ratio of the area of the parallelogram to the area of the triangle.

By "all lines" of a figure Cavaglieri probably understood the sum of these lines, i.e. an infinite quantity ("unbounded"), and hence only the *ratio* of these sums could be finite. Apparently (although Cavaglieri never stated this explicitly) the indivisibles are at equal distances from each other, but these distances appear nowhere.

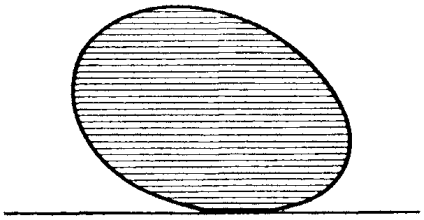

FIG. 110.

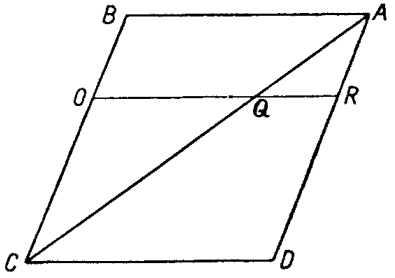

FIG. 111.

If we try to render Cavaglieri's idea in our customary terminology we may state that he uses the sum of the ordinates (or the sum of the values of function) without multiplying them by the increments of the abscissa (independent variable). Thus, the formulated proposition (taking a square with side a instead of the parallelogram, for simplicity, and reintroducing multiplication by the distance h between the indivisibles) can (of course, conditionally) be illustrated by means of the chain of relations

$$\frac{\sum OR}{\sum QR} = \frac{\sum a}{\sum x} = \frac{\sum ah}{\sum xh} = \frac{\int_0^a a\,dx}{\int_0^a x\,dx} = 2.$$

A further important step was made by Cavaglieri by establishing in *Geometry* the ratio of "all squares (lines OR) of the parallelogram" to "all squares (lines QR) of the triangle". In consequence of a long sequence of deductions it proved to be equal to *three*. In "experiment IV" he further compares "all cubes" and "all squares" (lines) of the parallelogram and the triangle: here the ratios prove to be *four* and *five*, respectively. Hence Cavaglieri also inferred the

validity of a similar law for a power with an arbitrary positive integral exponent m. In our symbolism this law can be written in the form

$$\frac{\int_0^a a^m\,dx}{\int_0^a x^m\,dx} = m+1,$$

hence the problem essentially consists in evaluating the integral

$$\int_0^a x^m\,dx = \frac{1}{m+1}\int_0^a a^m\,dx = \frac{1}{m+1}a^{m+1}.$$

Cavaglieri immediately applies his results to various squarings and cubings but derives them entirely independently of any applications. This generality of the formulation of the problem (as in the problem of evaluating a definite integral) constitutes great progress compared with Kepler's works in which only definite squarings and cubings were carried out.

219. Further development of the science of indivisibles. The evaluation of the integral $\int_0^a x^m\,dx$ by comparing it with the integral $\int_0^a a^m\,dx = a^{m+1}$ was also considered by other scholars. A French mathematician, Pierre Fermat (1601–1665), obtained Cavaglieri's general result somewhat earlier than the latter. We should also mention Blaise Pascal (1623–1662), a French mathematician, physicist and philosopher who wrote *Sum of Numerical Powers* (1654), and an English scholar, John Wallis (1616–1703), whose book *Arithmetic of Infinite Quantities* (1655) has already been mentioned. All these authors based their considerations on arithmetical reasoning and connected the evaluation with an investigation of the sum of m powers of consecutive positive integers. In the customary language the essence of the matter may be expressed as follows: if we subdivide the interval $[0, a]$ into n equal parts of length $h = a/n$ the ratio of the integral sums is

$$\frac{\sum_{i=1}^{n}\left(\frac{i}{n}a\right)^m h}{\sum_{i=1}^{n} a^m h} = \frac{\sum_{i=1}^{n} i^m}{n^{m+1}} \to \frac{1}{m+1} \quad \text{as} \quad n \to \infty.$$

Incidentally, the passage to a limit can be found in an explicit form only in the works of Wallis. All his reasoning is based on inductive methods.

In later papers, Fermat, dealing with the squaring of various "parabolas" $y^m = cx^n$ and "hyperbolas" $y^m x^n = c$, divides the figure under the curve into strips (just as we do) so small that they can be "set equal" to rectangles. The abscissae then form not an arithmetical but a geometric progression [cf. Sec. 184, *(2)*]. Thus Fermat was in a position to evaluate integrals of powers x^r with rational exponents $r = \pm n/m$ (except only the case $r = -1$ corresponding to the classical hyperbola).

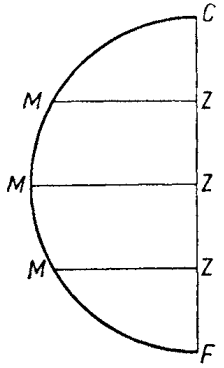

FIG. 112.

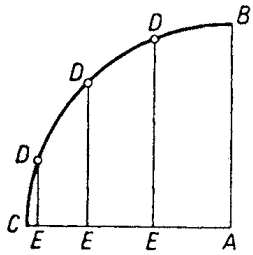

FIG. 113.

Pascal was close to the modern concept of the definite integral and he discovered the power of the (yet undiscovered) integral calculus. We have in mind his works which provide the solution of a number of problems announced by him in 1658; these problems were connected with the cycloid and sought to evaluate various areas, volumes, lengths of arcs and determine the locations of various centres of mass. These works were initially published incognito and were entitled *Various Discoveries of A. Dettonville in Geometry.*

[Pascal continued to employ "the language of indivisibles", but in an extensive "Forewarning" he elucidates in detail how this language should be understood. For instance, if the diameter of a semicircle (Fig. 112) is divided into an "unbounded" number of equal parts at points Z, and the ordinates ZM are drawn, by "the sum of the ordinates" one should understand "the sum of an

unbounded number of rectangles constructed of every ordinate and every very small equal part of the diameter", the sum "differing from the area of the semicircle by a quantity smaller than an arbitrary one". In Fig. 113 the arc BC of the circle is divided into an "unbounded" number of equal arcs at points D from which perpendiculars DE are drawn, the latter being called "sines". In this case, if we speak simply of the sum of sines DE, we mean by this the sum

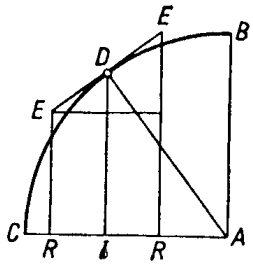

FIG. 114.

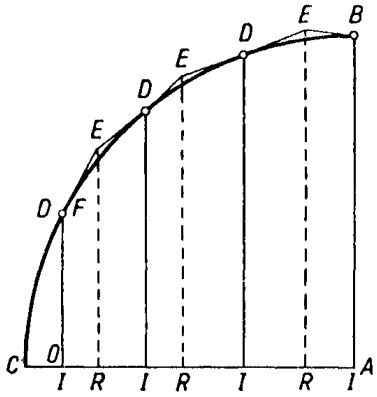

FIG. 115.

of rectangles constructed of every sine DE and of every rectifiable small arc DD, since these sines are generated by equal segments of the arc. In the examples given it is clear by which parts of the line the ordinates or sines should be multiplied; in other cases, however, the line should be explicitly indicated. Thus the independent variable set aside by Cavaglieri, who considered the sum of values of a function only, is here entirely clearly re-established: the values of the function are multiplied by the increments of the independent variable.

To give a specimen of the reasoning employed by Pascal for the evaluation of the required integrals, we quote a proposition from *The Treatise on Sines of Quarter of a Circle.* First of all the obvious lemma (see Fig. 114 which clearly indicates the notation) is established:

$$DI \times EE = RR \times AB. \tag{1}$$

The proposition itself states the following: the sum of sines of the arc BF (Fig. 115) is equal to the segment AO multiplied by the radius AB.

Replacing in (1) every tangent EE by the arc DD, and adding relations of this type, we obtain on the left the required "sum of sines" and on the right the sum of all RR or, equivalently, the line AO multiplied by AB. This completes the proof.

Now an interesting "Forewarning" follows in which Pascal tells the reader not to be surprised by the fact that "all distances RR are equal to AO and that every tangent EE is equal to every small arc DD, since it is well known that although this equality is *not true* when the set of the sines is finite, it *is true* when this set is unbounded".

To interpret this assertion in our language, let the radius $AB = 1$ and introduce the angle $\varphi = \sphericalangle BAD$; then it is equivalent to the relation

$$\int_0^{\varphi} \cos\varphi \, d\varphi = \sin\varphi .$$

The approach of Pascal to the solution of the considered problems is instructive; he first precisely enumerates in general form the types of integrals ("sums") required for the solution. Then he indicates how to evaluate them in the actual case under consideration; subsequently he completes the solution. We also mention various rather complicated integral formulae for the transformation of integrals ("sums") into other integrals; Pascal derives them from stereometric considerations and uses them with great skill.

220. Determination of the greatest and smallest quantities; construction of tangents. We now proceed to the early history of differential and integral calculus. The originator in this field was Fermat, who investigated both of the following problems, which are usually referred to the differential calculus: the determination of the greatest and smallest quantities and the construction of tangents: he was also the first to apply a method of an essentially infinitesimal nature to their solution.

Fermat's work *The Method of the Investigation of the Greatest and the Smallest Quantities* became known from his letters, beginning in 1629; it was partly published in 1642–1644 and fully published in 1679 posthumously.

The rule proposed by Fermat (without any justification) for the determination of the greatest and smallest quantities will be illustrated by means of one of the problems he investigated: it is required to cut a line AC (Fig. 116) at a point B in such a way that the body constructed on the square AB and the line BC has the greatest volume.

Denoting the known segment AC by B, and the unknown AB by A, we obtain for the greatest volume the expression $A^2(B-A)$†.

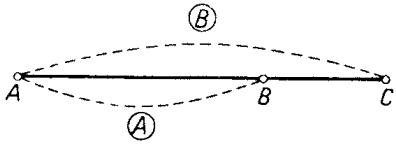

FIG. 116.

Substituting $A+E$ for A (Fermat used the letter E as a standard notation for the increment of the quantity A), we equate the two expressions (which are not in fact equal):

$$(A+E)^2(B-A-E) = A^2(B-A).$$

We now omit the terms common to the two sides and divide by the common factor E; then

$$2A(B-A) - A^2 + E(B-A-E) - 2AE = 0.$$

Finally, we disregard all terms which, after the above division, still contain the factor E. Hence

$$2A(B-A) - A^2 = 0 \quad \text{or} \quad 2AB = 3A^2.$$

According to Fermat's expression this is the "true" relation whereas the preceding ones were only "approximate" or "imaginary". From the last relation we find $A(=2B/3)$.

Using the functional notation the general form of "Fermat's rule" is as follows. To determine the quantity A such that the expression $f(A)$ has the greatest or smallest value, Fermat first writes down the "approximate relations"

$$f(A+E) = f(A) \quad \text{or} \quad f(A+E) - f(A) = 0,$$

† We employ throughout the standard algebraic notation, whatever notation the particular author may use.

whence, dividing by E, he obtains

$$\frac{f(A+E)-f(A)}{E}=0.$$

In this relation he disregards the terms still containing E, i.e. he sets $E=0$ (this is equivalent to passing to the limit $E\to 0$). Then we finally arrive at the "true" relation

$$\left[\frac{f(A+E)-f(A)}{E}\right]_{E=0}=0$$

or, in our notation, $f'(A)$; hence the required A is found [Secs. 100, 112].

Although Fermat did not say so, the quantity E plays the role of a very small (but not infinitesimal) increment of the independent variable A. The original relation $f(A+E)=f(A)$ expresses a kind of principle of cessation: at the instant when the quantity reaches its greatest or smallest value it ceases to change†.

In the same work Fermat indicates that his method is also applicable to the construction of the tangents to curves. Now, he denotes by A the subtangent and by E its increment (or decrement); making use of the equation of the curve he first constructs the "approximate" relation, applies the previous procedure and in consequence derives the relation for the determination of A.

Fermat's investigations are connected with rules given by other authors for the solution of the problems, rules which either simplify Fermat's or extend their domain of application. We mention, as an example, the method of constructing tangents given by Newton's teacher Isaac Barrow (1630–1677) in his *Optical and Geometrical Lectures* (1669–1670); he states that he follows "the advice of his friend" (apparently Newton).

Barrow introduces a standard notation for both the coordinates of the point M of the curve (Fig. 117) and for their increments, setting $AP=f$, $PM=m$, $NR=e$, $RM=a$; he regards these increments and the arc NM as "infinitely small". Connecting the coordinates $f-e$ and $m-a$ of the point N by the equation of the curve Barrow

† A similar principle had been formulated earlier, for example, by Kepler.

disregards all terms in the derived relation which do not contain either e or a (they in fact cancel each other), and also the terms of order higher than the first with respect to e and a ("since these terms are of no importance at all"). Here we encounter, for the first time in an explicit form, the principle of disregarding terms of a higher order of smallness (in Fermat's works it can only be suspected).

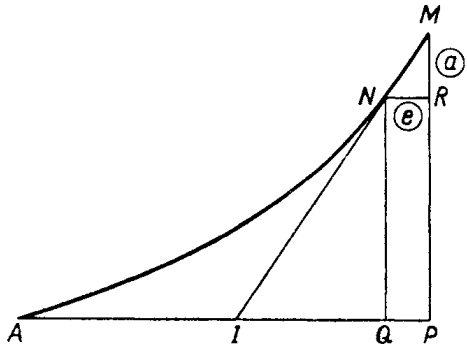

FIG. 117.

Now it is easy to find the ratio of a to e, which is the same as the ratio of the ordinate $PM = m$ to the subtangent $TP = t$. Equality of these two ratios follows from the similarity of the finite triangle TPM and the infinitesimal triangle NRM (in which, in view of the "infinite smallness", a "part of the curve" is replaced by a "part of the tangent").

Since then these similar triangles have been constantly used in the analysis of infinitesimals. Subsequently Leibniz called them "characteristic"†.

221. Construction of tangents by means of kinematic considerations. The French mathematician Jules Personne de Roberval (1602–1675) and the Italian physicist and mathematician Evangelista Torricelli (1608–1647), independently of each other and almost simultaneously (their investigations were first published in 1644), conceived the idea of using kinematical considerations in the construction of

† Incidentally, according to his statement, the idea of the infinitesimal "characteristic" triangle was adopted by him not from Barrow, but from Pascal (see Fig. 114).

tangents to curves. If it is possible to represent a curve as the trajectory of a moving point whose motion is the resultant of two simpler movements for which the directions and magnitudes of the velocities are known, the direction of the compound motion and, therefore, the direction of the tangent to the curve, can be determined in accordance with "the parallelogram law".

As an example we present Torricelli's solution of the problem of constructing a tangent to a parabola. He uses the kinematical considerations of his teacher Galileo which we, for brevity (departing from the original), give in the language of analytic geometry. Suppose that the point is initially located at O (Fig. 118) and falls freely with

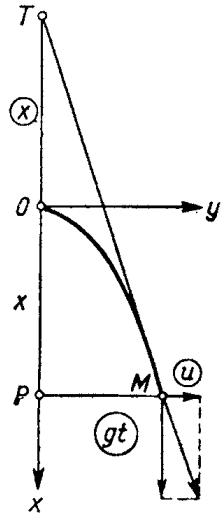

FIG. 118.

acceleration g (and hence with velocity gt, t denoting the time) along a vertical straight line which itself is displaced horizontally with constant velocity u. Then using the notation of the figure, at the instant t we have

$$x = \tfrac{1}{2} gt^2, \quad y = ut.$$

Hence, eliminating t we obtain $y^2 = 2(u^2/g)x$. Thus the trajectory of the point is a parabola (which can be identified with an arbitrary

parabola by a suitable choice of u). The ratio of the vertical and horizontal velocities is

$$\frac{gt}{u} = \frac{gt^2}{ut} = \frac{2x}{y}.$$

Hence, taking into account the similarity of the triangles we find that the tangent intersects the axis of the parabola at a distance x behind its vertex [cf. Sec. 210, *(2)*].

We have considered this example in detail since, in order to construct the tangent, we decomposed the motion along the curve into composite motions along the horizontal and vertical directions.

Subsequently Barrow, extending this concept, represented the motion along an arbitrary curve as composed of two motions—a horizontal one (which can always be regarded as uniform) and a vertical one. Then the location of the tangent TM (Fig. 118) is determined by the ratio of the segments TP and PM, which is equal to the ratio of the velocity of the "fall" to the velocity of the "side motion".

222. Mutual invertibility of the problems of construction of tangent and squaring. The tenth and eleventh of Barrow's *Lessons on Geometry* are of major importance and interest; in these lessons the construction of tangents is connected with squaring. From a large number of relevant theorems we consider here Theorem XI from Lesson X and Theorem XIX from Lesson XI in which, for the first time in the history of analysis of infinitesimals, the two basic problems of differential and integral calculus in geometric form are compared directly, namely the construction of the tangent and the squaring of a curve. In the analytic language, using the customary notation the above theorems can be stated as follows:

I. If $y = \int_0^x z\,dx$, then $\frac{dy}{dx} = z$.

II. If $z = \frac{dy}{dx}$, then $\int_0^x z\,dx = y$ (it is assumed that for $x = 0$ we have $y = 0$).

To demonstrate the nature of Barrow's work we give briefly the statement and proof of the second theorem.

An arbitrary curve AB is given (Fig. 119). Let MT be the tangent to it at the point M. The second curve KL is defined by the condition $FZ : R = FM : TF$ where R is a given segment ($= DH$). Then the area $ADLK$ is equal to the product $DB \times R$.

To prove the statement we take on the curve AB an "infinitely small segment MN" and we draw the lines shown in the figure. Now we know that

$$MO : NO = FM : TF = FZ : R,$$

whence

$$NO \times FZ = MO \times R \quad \text{or} \quad GF \times FZ = ES \times EX.$$

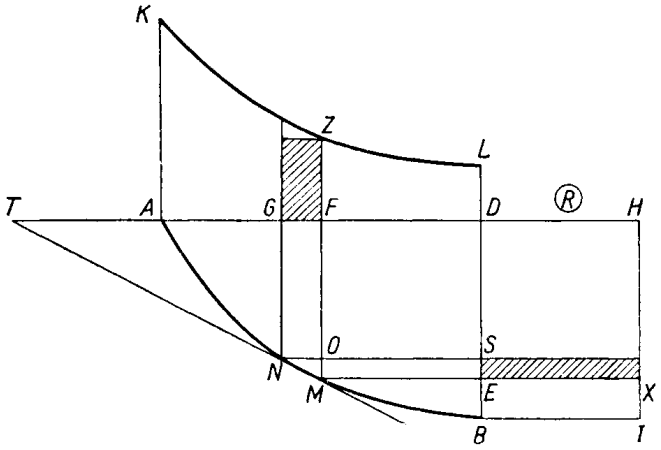

FIG. 119.

"But since all rectangles $GF \times FZ$ differ by an arbitrarily small amount from the area $ADLK$ and all the corresponding rectangles $ES \times EX$ constitute the rectangle $DHIB$ the statement is sufficiently clear."

Setting $AF = x$, $FM = y$, $FZ = z$ and $R = 1$, by the condition defining the second curve we have

$$\frac{z}{1} = \frac{FM}{TF} = \frac{dy}{dx},$$

and the conclusion of the theorem is equivalent to the statement

$$\int_0^x z\,dx = y \times 1 = y.$$

It would be in vain, however, to seek in Barrow's work even a simple comparison of these two theorems (they are separated by many other theorems); moreover, they are rarely used. This was the influence of the geometric language used by Barrow, who did not possess the general ideas which could have revealed the essence of the matter and paved the way to extensive applications.

223. Survey of the foregoing achievements. We now summarize the achievements of the seventeenth century in "the analysis of infinitesimals", up to the time of Newton and Leibniz.

The main results concerned the subject now referred to as the integral calculus. Not only were a great number of particular results derived, concerning the squaring, cubing, rectification of curves, developing of surfaces and determination of the centre of mass, but also the connection was established between such problems, which were traditionally reduced to the first one—the squaring. In the papers of Cavaglieri, Pascal and others the definition of the definite integral was gradually set up. A number of simple integrals were in fact evaluated, mostly in geometric form but sometimes purely arithmetically (Fermat, Pascal, Wallis); various relations were found which transformed certain integrals into others (Fermat, Pascal, Barrow).

In the subject now referred to as the differential calculus Fermat announced a unified method of infinitesimal nature for the solution of problems concerned with the determination of the greatest and smallest values and the construction of tangents. His investigations were continued by a number of other authors. However, at this point they did not succeed in separating out the basic concepts which constitute the essence of the problem. The attempts of Roberval and Torricelli were exceptional; prior to Barrow's work they tried to solve the problem of constructing a tangent to a curve on the basis of kinematical considerations (which later influenced Newton's concepts).

Finally, as we have seen, Barrow succeeded in partially discovering the connection between the problems of the two groups.

Thus the ground for the new calculus was prepared but the calculus itself still had not appeared. But at the same time, as Leibniz later said, "after these successes of the science one thing

only was lacking—the Ariadne thread in the labyrinth of problems—an analytic calculus following the pattern of algebra". It was necessary first of all to establish, in a general form, the basic concepts of the new calculus and their connexion. Then, introducing an appropriate symbolism, it was necessary to create a standard procedure or algorithm for the computations. This was accomplished by Newton and Leibniz, independently and in different ways†.

A survey of their work on the analysis of infinitesimals will be preceded by the following remark concerning the concept of an "infinitesimal". At that time, and for a long time afterwards, an infinitely small quantity was tacitly regarded as a static, i.e. invariable, quantity, distinct from zero, its absolute value being smaller than any finite quantity. This concept of an "actual" infinitesimal, under our concept of number and space, is contradictory and of a mystic nature. It is in contrast to the (later customary) concept of a "potential" infinitesimal as a variable quantity which in the course of its variation only becomes (again in absolute value) smaller than any finite quantity. The transition from one concept of infinitesimal to the other encountered great difficulties since it required a clear understanding of the concept of the passage to a limit. The reader will see in the works of Newton and Leibniz the struggle between these two concepts. We now consider the work of these two authors.

§ 2. Isaac Newton (1642–1727)

224. The calculus of fluxions. The basic work of Newton in which the calculus is presented is the treatise *The Method of Fluxions and Infinite Series*. It was written about 1671 (its basic concepts were formed earlier) but was not published until 1736, after Newton's death. The variable quantities were called "the fluents" (i.e. "current" quantities) by Newton and denoted by the last letters of the Latin alphabet u, y, z, x; they were regarded as increasing (decreasing) with time. Their velocities of increase were called "fluxions" and denoted by the same letters with dots: $\dot{u}, \dot{y}, \dot{z}, \dot{x}$. Thus for Newton

† We shall not consider the unjustified controversy, which arose later, concerning the priority of the discovery of the new calculus.

the velocity was an obvious concept which did not require a definition and it served to define the fluxions, i.e. in our language, the derivative of the fluent with respect to time†.

It is true that Newton stipulates that time should not be used in the literal sense—for "time" any quantity may be taken, say x, which increases uniformly with time, for instance a quantity such that $\dot{x} = 1$. However, it should be borne in mind that all fluents depend on this "time", i.e. on one universal independent variable. Thus, neither functions of several variables nor partial derivatives were considered by Newton.

The first basic problem was then formulated by Newton as follows: "To determine the relationship between fluxions in accordance with a prescribed relationship between the fluents."

This problem is more general than the simple calculation of a fluxion in terms of a given fluent. Newton, however, solves this problem directly for algebraic equations only. For instance, he takes the equation

$$x^3 - ax^2 + axy - y^3 = 0. \tag{1}$$

The rule proposed by Newton is the following: every term containing a power of x is multiplied by the exponent of the power x and one of the factors x is replaced by $\dot{x}$; similarly, every term containing a power of y is multiplied by the exponent of y and one of the factors y is replaced by $\dot{y}$; the sum of all the terms found in this way is set equal to zero. In the above example we obtain

$$3x^2\dot{x} - 2ax\dot{x} + ay\dot{x} + ax\dot{y} - 3y^2\dot{y} = 0.$$

It is readily observed that this rule can be extended to the general case of an algebraic equation containing an arbitrary number of fluents. When fractions or roots are present Newton employs an indirect method. Consider the equation

$$x^3 - ay^2 + \frac{by^3}{a+y} - x^2\sqrt{(ay + x^2)} = 0.$$

† Although Newton's symbolism is not used any more, in mechanics and physics it is still customary to denote derivatives with respect to time by dots.

Setting

$$\frac{by^3}{a+y} = z \quad \text{and} \quad x^2\sqrt{(ay+x^2)} = u,$$

Newton reduces it to the equation

$$x^3 - ay^2 + z - u = 0,$$

to which the above rule is applicable:

$$3x^2\dot{x} - 2ay\dot{y} + \dot{z} - \dot{u} = 0.$$

$\dot{z}, \dot{u}$ can in turn be determined from the above relations by an application of the same rule to the equations

$$az + yz - by^3 = 0, \quad ax^4y + x^6 - u^2 = 0.$$

In giving a proof of the rule Newton introduces a new concept: the "moments" of the current quantities. They are "those infinitesimal parts of them arising from the addition of infinitesimal parts of time, the very quantities increasing continuously". These moments are proportional to the velocities with which the quantities vary, i.e. the fluxions. Introducing an infinitesimal quantity o (this is not zero but an "actual" infinitesimal increment of time) Newton denotes the moments of quantities by $\dot{u}o, \dot{y}o, \dot{z}o, \dot{x}o$ (Leibniz's differentials).

The proof itself is carried out by Newton for the above example, in essence, repeating Fermat's procedure. Substituting in relation (1) $x + \dot{x}o$ instead of x and $y + \dot{y}o$ instead of y, he subtracts (1) term by term, divides by o and finally disregards terms which still contain o; his explanation is as follows: "since we have assumed o to be an infinitesimal quantity ..., the terms multiplied by it can be regarded as nought in comparison with other terms." Neither this principle nor the rule itself is formally new, but the essential new feature is that the result is stated for a fluent of arbitrary nature, independently of any particular problems.

Subsequently Newton also introduced the fluxion of a fluxion, i.e. the second fluxions $\ddot{u}, \ddot{y}, \ddot{z}, \ddot{x}$ and even fluxions of higher orders.

Newton first applied his calculus of fluxions to problems mentioned frequently above.

"To determine the greatest and smallest value of a quantity."

First he states the principle of cessation: "When the quantity has the greatest or the smallest of all possible values, then at this instant it does not flow either forward or backwards." Hence the following rule follows: find the fluxion and equate it to zero. Then, as Newton emphasizes, the relation determining the fluent may also contain irrational quantities, which was not allowed by the rules published before.

"To construct the tangent to a curve."

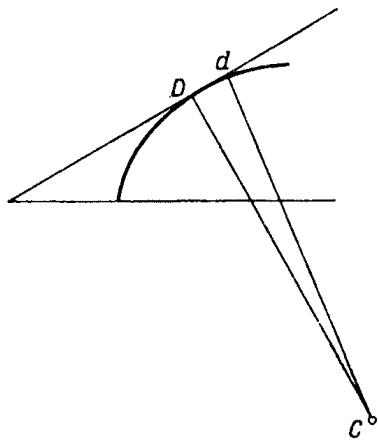

FIG. 120.

In the basic case when the equation between the Cartesian coordinates x, y of a variable point of the curve is known, Newton's reasoning is similar to that of Barrow [Sec. 221], but with the infinitesimal increments (decrements) e and a replaced by the moments $\dot{x} \cdot o$ and $\dot{y} \cdot o$, and hence (using the notation of Fig. 117)

$$PM : TP = \dot{y} : \dot{x};$$

by the given rule the ratio of fluxions is determined from the equation of the curve. Newton also examined a number of other methods of constructing a tangent, corresponding to other forms for the equation of the curve.

The formulation of the following problem was entirely new:

"To determine the magnitude of the curvature of a curve at a point."

After having stated the problem Newton added: "There exist few problems in the theory of curves, which are more elegant and would better reveal their nature."

The definition of the concept of curvature is not given. The curvature of a circle is the same at all points, and is proportional to the diameter. The curvature of the curve at a point D (Fig. 120) is identical with the curvature of that circle which touches the curve nearest this point (in fact, Newton regarded the curve and the circle as coinciding over an infinitesimal arc Dd. If C is the centre of this circle (the "centre of curvature"), then at this point there intersect the two infinitely near normals CD and Cd of the curve. Newton derived a formula for the radius of the circle (the "radius of curvature") which is different only in form from the customary one.

225. The calculus inverse to the calculus of fluxions; squaring. Following the first basic problem, Newton in the *Method of Fluxions and Infinite Series* also formulates the second inverse problem:

"To determine the relationship between the fluents in accordance with the prescribed relationship connecting the fluxions."

In this form it is (as we should now say) the problem of the integration of an ordinary differential equation; it is a more general and difficult problem than the direct determination of fluents in terms of fluxions, i.e. the determination of the primitive. Here we do not examine the above general problem (Newton himself solves it mostly by applying infinite series) and we shall deal only with the problem of the determination of the primitive which was always treated by Newton geometrically—as a problem of squaring a curve.

The basis is formed by the fundamental proposition that (in our customary terminology) the derivative of a variable area with respect to the abscissa is the ordinate and therefore the area itself is, for the ordinate, the primitive function [cf. Sec. 156].

It is of interest to examine the proof of this proposition, which was given in an earlier work† of Newton, before the creation of

† *Analysis by Means of Equations with an Infinite Number of Terms*; see *Mathematical Works of Newton*. This work was written as early as 1666–1667 but was not published until 1711.

the method of fluxions. Attempting to establish that the area z of the curve $y = ax^{m/n}$ (measured from the point at which $y = 0$) is given by the formula $z = [an/(m+n)]x^{(m+n)/n}$, Newton used the inverse procedure and derived from the expression for the area the expression for the ordinate. He began by considering the particular example for which $z = 2x^{3/2}/3$, so $y = x^{1/2}$; we re-establish the relevant reasoning, which is of an entirely general nature.

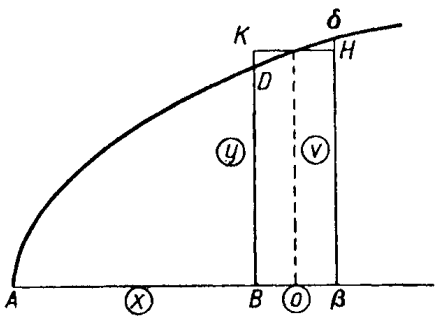

FIG. 121.

Thus, let (Fig. 121) $AB = x$, $BD = y$ and the area $ADB = z$. Set $B\beta = o$ (here o does not denote the increment of time, as in the theory of fluxions) and define $BK = v$ in such a way that the rectangle $B\beta HK$ has the same area ov as the figure $B\beta\,\delta D$; then $A\beta = x + o$ and $A\delta\beta = z + ov$.

Substituting these expressions for x and z in the relation $2x^{3/2}/3 = z$ or $4x^3/9 = z^2$, after the usual procedure of disregarding the common terms and dividing by o, we arrive at the relation

$$\frac{4}{9}(3x^2 + 3xo + o^2) = 2zv + ov^2.$$

"If now"—continues Newton—"we assume that $B\beta$ decreases infinitely and vanishes or that o is zero, then v and y become equal and the terms with the factor o vanish." Hence it is easy to obtain the required result $y = x^{1/2}$.

Since, in fact, v is the ratio of the increment of the area ($= ov$) to the increment of the abscissa ($= o$), and the statement that v becomes equal to the ordinate when o decreases infinitely is not connected with the particular problem considered, this is essentially

the proof [cf. Sec. 156] of the above proposition. Observe that $o = B\beta$ is here meant rather as an infinitesimal, and a definite hint on passing to the limit may be assumed.

Newton proceeded differently in the *Method of Fluxions*. Besides the variable curvilinear figure ADB he also considered the variable rectangle $ACEB$ with height $AC = 1$ (Fig. 122). Both areas "are generated" by the motion of the straight lines BD and BE, respectively. "Then the ratios of the increments of these areas†

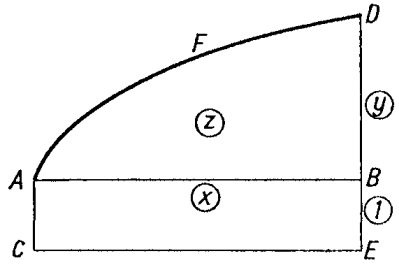

FIG. 122.

and their fluxions are always the same as those of the corresponding lines." Using the previous notation (and taking into account that the area of the rectangle is x), we have

$$\frac{\dot{z}}{\dot{x}} = \frac{y}{1} \quad \text{or} \quad \dot{z} = y\dot{x}.$$

Assuming that $\dot{x} = 1$, we simply obtain $\dot{z} = y$. Both these results are constantly used by Newton.

Now it is easy to solve the following problem:

"To find an arbitrary number of curves the areas of which are representable by means of a finite equation."

That is, given an arbitrary equation connecting x and z, it is required to find the relation between x and $\dot{z} = y$; in this way we obtain a curve the area of which has a prescribed form, in terms of the abscissa (or, generally, they are connected by a known equation).

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† Now, apparently, "actual" infinitesimals.

Subsequently Newton stated the following problem:

"To find an arbitrary number of curves the areas of which are connected with the area of a prescribed curve by a finite equation."

Briefly, an integral is reduced to another form by means of a substitution but the operation is carried out (as above) in an inverse order: a function is sought the integral of which could be expressed in terms of the given integral by the given equation using the given substitution.

Making use of these two devices Newton constructed extensive "catalogues" of curves the squaring of which is performed directly, or (by means of indicated substitutions) it is reduced to the squaring of an ellipse or hyperbola ("the areas of which may be considered as known in a way"). The reduction to the squaring of conic sections meant, in fact, using the simplest transcendental functions—the logarithmic and the inverse trigonometric functions, which at that time were not yet introduced into analysis.

Another work of Newton, *A Consideration on the Squaring of Curves* written soon after the *Method of Fluxions* and published in 1704†, was devoted to the evaluation of squarings. In this work Newton also considers expressions of a more complicated form, for instance,

$$z^{\theta}(e+fz^{\eta}+gz^{2\eta}+\dots)^{\lambda}(a+bz^{\eta}+cz^{2\eta}+\dots),$$

where θ, λ, η are rational exponents. As a particular case let us note the determination of the binomial integrals, i.e. the determination of the primitive function for the expressions of the form

$$z^{\theta}(e+fz^{\eta})^{\lambda}.$$

Incidentally, more details were supplied by Newton in a letter to Leibniz (1676): he knew that the squaring might be performed algebraically if $(\theta+1)/\eta$ was a positive integer or $[(\theta+1)/\eta]+\lambda$ was a negative integer [cf. Sec. 169].

As regards applications of the calculus of squarings, in *Method of Fluxions* Newton clearly stated that the tables of areas of curves may also be used for the determination of quantities of other kinds in accordance with known fluxions. The following problem is an example.

† See *Mathematical Papers of Newton.* The introduction and other parts of *A Consideration* bear the traces of a later treatment.

"To determine the lengths of curves."

The problem reduces to the determination of the arc $t = QR$ (Fig. 123) in terms of its fluxion $\dot{t} = \sqrt{(\dot{z}^2 + \dot{y}^2)}$ [cf. Sec. 202, (5)], where $z = MN$ and $y = NR$ are the abscissa and the ordinate of the variable point R of the curve $\dot{y}$. The formula for $\dot{t}$ follows directly from the consideration of the right-angled triangle RSr the sides of which are the "moments" of the quantities z, y, t.

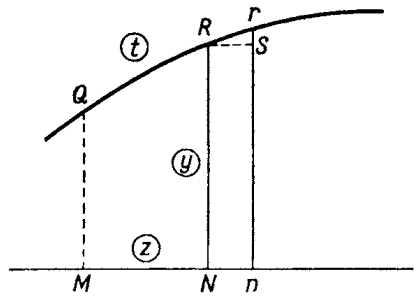

FIG. 123.

226. Newton's *Principles* and the origin of the theory of limits. *Mathematical Principles of Natural Philosophy was the work which,* more than any other, made Newton famous; it was published in 1686–1687. It contains the foundations of the whole of mechanics, and of celestial mechanics in particular.

Newton states in one of his letters that he found the most important propositions of his *Principles* by the method of fluxions. However, in the exposition itself this statement is not justified; according to the example of ancient philosophers, proofs of the propositions were given in the language of synthetic geometry.

Nevertheless, the *Principles* also contains essential results from the methodological point of view. The first part of the first book ("On the motion of bodies") is devoted by Newton to a special theory of limits, the title being "Method of first and last ratios".

The "first ratios" or the "last ratios" of two quantities are their limiting ratios. The first term is employed by Newton to denote the ratio of two "generated" (infinitesimal) quantities, while the second is used both to denote the ratio of "vanishing" quantities and the ratio of finite or even infinitely large quantities. Newton

even speaks of "the first sum of generated quantities" or "the last sum of vanishing quantities". It is important to note that all these concepts are not defined and their meaning can only be elucidated from the method of application. The special terminology of Newton is connected with the concept of a variable reaching its limit, which is its "last" ("first") value.

The whole of Newton's theory of limits consists of eleven lemmas of a geometric nature. As Newton indicated in the "Instruction" following the lemmas, the latter are given to shorten the proofs. The same result could also be attained by means of the method of indivisibles, but this would be "less geometrical". "Therefore"—continued Newton—"when throughout the following exposition I regard some quantities as if composed of constant parts ... it should be understood that they are not indivisibles but vanishing divisible quantities, not sums and not ratios of finite parts but the last sums and the last ratios of vanishing quantities...." And further, "if in what follows for the sake of simplicity I speak of very small or generating or vanishing quantities, one should not understand by these quantities of a definite magnitude but as infinitely decreasing". Thus he states here a point of view essentially similar to the modern one: instead of "actual" infinitesimals "potential" infinitesimals are introduced, as well as the limits of their sums and ratios.

227. Problems of foundations in Newton's works. We see that Newton's viewpoint in the problems of the foundations of his calculus underwent a considerable development over a period of twenty years.

In his *Method of Fluxions* representing his old viewpoint the "moments" of quantities are clearly the "actual" infinitesimals and the increase of a quantity is reduced to their successive addition. The concept of disregarding infinitesimal quantities in comparison with finite ones is freely employed.

In his *Principles* Newton dissociates himself from the viewpoint of indivisibles. In the introduction to *Squaring of Curves,* which was written later, he states: "I regard the mathematical quantities not as composed of minute parts, but as described by a continuous motion." From a remark in the second edition of the *Principles* (1713) it follows that "in the method of the generation of quantities"

Newton perceives the principal difference between his method and Leibniz's method. The theory of limits which is found in the *Principles* in a rudimentary form constitutes considerable progress in the problem of the foundations of the new analysis. Subsequently, in the above-mentioned introduction to *Squaring of Curves* even the derivation of the fluxions of x^n is connected with the consideration of "the last ratio" of two vanishing quantities, i.e. essentially with a passage to the limit.

However, Newton did not carry out his viewpoint to the end. As early as in the second volume of *Principles* he again introduced the obscure concept of the "moments" of quantities, i.e. their instantaneous increments or decrements".

Concerning these "moments", a number of simple propositions are established (it should be said that they had already been published in an equivalent form by Leibniz). Here is an example: if the moments of quantities A and B are a and b, then the moment of the product AB is $Ab + Ba$. It is noticeable that in the proof of this proposition Newton does not use the naturally appearing relation

$$(A + a)\ (B + b) - AB = Ab + Ba + ab,$$

since then he would have had to disregard the term ab as compared with the other terms (which is precisely Leibniz's method), but he employs a device, as shown in the relation

$$(A + \tfrac{1}{2}a)\ (B + \tfrac{1}{2}b) - (A - \tfrac{1}{2}a)\ (B - \tfrac{1}{2}b) = Ab + Ba,$$

which leads at once to the required result but does not follow from the essence of the problem.

Thus Newton's attempt to create by the "method of first and last ratios" a sound foundation for the new calculus was not consistent. It was developed and completed in the papers of mathematicians of the nineteenth century after the passing of a hundred years [Sec. 233].

§ 3. Gottfried Wilhelm Leibniz (1646–1716)

228. First steps in creating the new calculus. Unlike Newton, Leibniz left an enormous number of dated manuscripts, making it possible to establish the order of development of his ideas.

In one of the manuscripts dated 1675 we first encounter the sign $\int$; Leibniz wrote "it will be convenient to write $\int$ instead of *all*, and $\int l$ instead of *all l*, i.e. instead of the *sum of l*" (where l denotes a line). Soon the sign of the difference d was introduced. Only gradually did Leibniz write dx under the sign of $\int$.

During the years 1676–1677 Newton and Leibniz twice exchanged letters (through a third person). Newton states in them his results on expansions into infinite series and on squarings. Mentioning a treatise (apparently he meant *Method of Fluxions*) Newton informs Leibniz that he is in possession of a method which makes it possible not only to solve problems for tangents or for the greatest or smallest quantities, but also facilitates the computation of squarings; however, he concealed the method itself. Leibniz immediately answered by describing his own method, but he confined himself to the differential calculus only.

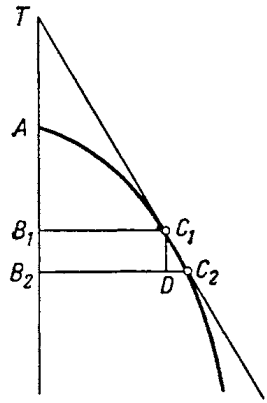

FIG. 124.

The ratio of the "segment TB_1 (Fig. 124) to the ordinate B_1C_1† is the same as C_1D (the difference of the two abscissae AB_1, AB_2) to DC_2 (the difference of two ordinates). ... It follows that the determination of tangents is equivalent to the determination of the differences of ordinates for equal differences of the abscissae. Consequently, if we denote by dy the difference of two adjacent values

† It should henceforth be borne in mind that Leibniz usually counts the abscissae along vertical and ordinates along horizontal lines.

of y and by dx the difference of two adjacent values of x, then evidently $d(y^2)$ is $2y\,dy$, $d(y^3)$ is $3y^2$, etc." For instance,

$$dy^2 = (y + dy)^2 - y^2,$$

or omitting the cancelling terms and the square $(dy)^2$ "according to the foundations known from the method of the greatest and the smallest"† we have $d(y^2) = 2y\,dy$.

Further, Leibniz gives formulae for the differentiation of a product and a root (regarding a root as a power); he differentiates even more complicated functions and emphasizes that "it appears in a most curious and convenient way that dy and dx are always outside the irrational term".

229. The first published work on differential calculus. Not until 1684 was the first memoir of Leibniz published, under the long title, *New method of the greatest and the smallest, and also tangents, for which neither fractional, nor irrational quantities are an obstacle, and a special kind of calculus.*

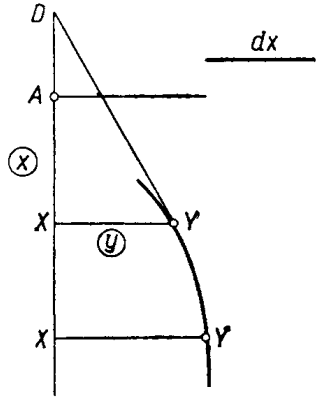

FIG. 125.

Here, initially, Leibniz tried to avoid infinitesimals and with respect to the "differences" (*differentia*) or "differentials" (*quantitas differentialis*) of variable quantities, he assumed a different viewpoint from that in the letter to Newton referred to above. Suppose that (Fig. 125) YY is an arbitrary curve, Y a variable point on it with

† This is a hint concerning the works of Fermat and others who solved the problem of the determination of the greatest and the smallest.

abscissa $AX = x$ and ordinate $YX = y$. Leibniz used dx to denote an arbitrary segment. If YD is a tangent to the curve at the point Y, then the segment for which the ratio of it to dx is the same as that of y to the (subtangent) XD is called dy.

Thus, unlike Newton, for whom the initial concept was the velocity, Leibniz's initial concept was the tangent.

Next Leibniz announced (without proof) "the rules of calculus" concerning the differentiation of a constant, sum, difference, product, quotient, power, root†. "If we knew, say, the algorithm of this calculus, which I call differential, then ... we could determine the greatest and the smallest and also the tangents, without having to eliminate fractions or irrationalities ... as it was necessary to do in making use of methods known so far." Considering a proof of the above, it is necessary to take into account that $dx, dy, \ldots$ may be regarded as proportional to "the instantaneous increments or decrements of $x, y \ldots$", respectively. Thus, in the end, the problem is reduced to the infinitesimals, as in the letter to Newton mentioned above.

Leibniz indicated that the greatest or the smallest ordinate is determined from the condition that the tangent should not be inclined in either direction, i.e. by the condition that $dy = 0$; at this instant the ordinates "neither increase nor decrease but are at rest". He distinguished between the greatest and the smallest values according to whether the curve is directed towards the axis by its concavity or convexity, and this is indicated by the sign of $d\,dy$. Finally, he investigated points of inflection (inaccurately, however).

Leibniz also solved a number of problems by his method, including the celebrated problem which exercised Fermat and other scholars of the seventeenth century—for instance, what should be the path of light from a point C in one medium to a point E in another medium (Fig. 126) in order that the path is covered in the shortest time? Leibniz introduced "the densities" h and r of the media (in the sense of "resistance encountered by the light in them") and seeks the point F on the straight line SS representing the plane of division

† In some of them double signs are encountered, since the subtangent is provided with no sign.

between the media, such that the path *CFE* "is the easiest of all possible paths", i.e. such that the quantity

$$w = CF \cdot h + FE \cdot r$$

is the smallest. Using the notation of the figure

$$w = hf + rg = h\sqrt{[(p-x)^2 + c^2]} + r\sqrt{(x^2 + e^2)}.$$

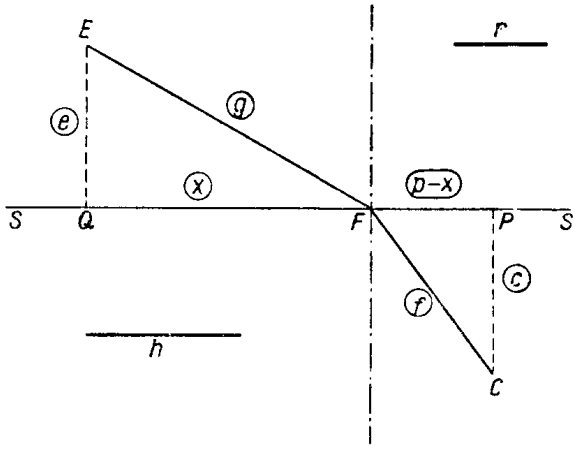

FIG. 126.

The required x is determined from the condition $dw = 0$ or

$$\frac{h(p-x)}{f} = \frac{rx}{g},$$

which can be written in the form

$$\frac{p-x}{f} : \frac{x}{g} = r : h.$$

It is readily observed that this expresses the following familiar law of physics: the sines of the angles of incidence and refraction are proportional to the optic densities of the two media. "Other very scholarly men"—concluded Leibniz—"were forced to use complicated methods to obtain the result which a man experienced in this calculus is able to carry out in three lines."

230. The first published paper on integral calculus. In 1686 Leibniz published a memoir *On Essential Geometry and Analysis of In-*

divisibles and Infinite Quantities where, for the first time, the sign $\int$ is encountered (in the form of a letter *s*).

First, he investigated a theorem of Barrow. If by y, x and p we denote the abscissa, ordinate and subnormal, then $p\,dy = x\,dx$ (this can easily be verified by making use of the infinitesimal "characteristic" triangle with the sides dy and dx). "If we convert this difference (differential) equation into a sum equation, then $\int p\,dy = \int x\,dx$. But from the results given by me in my method of tangents it follows that $d(x^2/2) = x$; consequently we also have conversely $x^2/2 = \int x\,dx$ (since the sums and the differences, or $\int$ and d, are mutually inverse, as in the ordinary calculus of powers and roots)." Hence $\int p\,dy = x^2/2$ which constitutes the contents of Barrow's theorem.

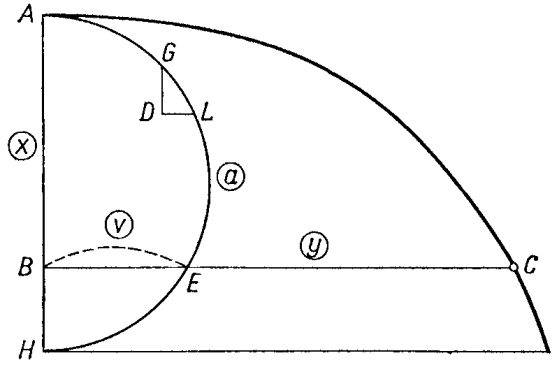

FIG. 127.

Leibniz emphasizes that his calculus makes it possible to express by means of equations also "the transcendental", i.e. non-algebraic, lines, for instance, the cycloid. We now give the relevant part of the memoir together with the explanations given by Leibniz himself in his letters. Figure 127 represents a semicircle and half of an arc of a cycloid; suppose that the radius of the circle is unity, $AB = x$, $BE = v$, $BC = y$, $AE = a$, $GD = dx$, $DL = dv$. Then, according to a familiar theorem of geometry, $v = \sqrt{(2x - x^2)}$ and consequently

$$dv = \frac{1-x}{\sqrt{(2x-x^2)}}\,dx;\quad GL = \sqrt{[(dx)^2 + (dv)^2]} = \frac{dx}{\sqrt{(2x-x^2)}},$$

$$a = \int \frac{dx}{\sqrt{(2x-x^2)}}.$$

Since, by the definition of a cycloid, $EC = a$ and $y = a + v$, we have

$$y = \sqrt{(2x - x^2)} + \int \frac{dx}{\sqrt{(2x - x^2)}}.$$

"This equation expresses perfectly the relationship between the ordinate y and the abscissa x and from it all properties of the cycloid can be derived." (For instance, by differentiation we can easily obtain the familiar construction of the tangent or normal to a cycloid.) Thus, for Leibniz, integration was a way of constructing transcendental functions, which by another method he could neither investigate nor denote.

At the end of the memoir Leibniz made an important remark stating that we should not disregard the factor dx in the integrand, since that would prevent the transformation of one figure into another. It is clear that he meant here a transformation of the variable, making it possible to reduce one squaring to another, and the last operation is in fact simplified by the presence of the factor dx.

Thus, for Leibniz, the fundamental concept in the integral calculus is the sum of "actual" infinitesimal rectangles $y\,dx$ (later, following the example of the Bernoulli brothers, he called this concept the integral); on the other hand, Newton, as we have already seen, took as his foundation the concept of a primitive function. For the purpose of applications, the point of view of Leibniz is more convenient, although he reduced the evaluation of the integral to the determination of the primitive function.

231. Further works of Leibniz. Creation of a school. The contents of the numerous papers and notes of Leibniz and also his correspondence with outstanding mathematicians of that time, were very diverse. They contain, first of all, a further development of his calculus. Some of the relevant problems have already been mentioned in the preceding chapters: the differentiation of power-exponential expressions [Sec. 85, (5)], a formula for the differentials of higher orders of a product [Sec. 98], and the decomposition of rational fractions into simple fractions for the simplification of integration [Sec. 166]. Other papers of Leibniz are connected with the expansion of functions into infinite series, or belong to more advanced topics of analysis (which will be encountered in the second volume). Besides constructing

the apparatus of analysis, Leibniz dealt with its applications, particularly in the field of "differential geometry". He frequently suggested to his contemporaries various problems, and, conversely, solved problems stated by others.

Of special importance was the fact that Leibniz created a school, outstanding members of which were the Bernouilli brothers, Jacob (1654–1705) and Johann (1667–1748), and also Gullaume François de l'Hôpital (1661–1704), the author of the first text book on differential calculus. The creation of the school was facilitated by the scientific enthusiasm of Leibniz, and the continuous publication of his papers and his scientific correspondence.

We should also not underestimate the convenience of the notation introduced by him; these are most suitable for geometrical and mechanical investigations (it is not without reason that Leibniz's notation has been preserved until now). The expedient symbolism undoubtedly assisted in creating the algorithm of which Leibniz dreamt from the very beginning. This algorithm gradually became common property.

232. Problems of foundation in Leibniz's works. In this respect Leibniz encountered major difficulties and continued to try to justify the new analysis until his death.

The "actual" infinitesimals are the foundation of both differential and integral calculus. Concerning the former, Leibniz [Sec. 229] still attempted to replace infinitesimal differences by proportional finite quantities; besides infinitesimals, as a characteristic triangle he considered a proportional finite triangle. But to derive his formulae he still cannot do without infinitesimals, and without the principle of disregarding infinitesimals of higher orders.

In reply to the attacks of the critics of the new calculus Leibniz proposed replacing "the infinitesimals" by quantities "incomparably small"; for example, a particle of dust as compared to the globe, or the globe as compared to the heaven. Moreover, in other papers Leibniz emphasizes that, by an infinitesimal, he by no means implies "a quantity very small, but always constant and defined"; this quantity need only be sufficiently small in order that the error be smaller than any indicated quantity. This may be regarded as a hint to a compromise with the idea of "potential" infinitesimals.

The possible solution Leibniz saw, regarding the infinitesimals as "fictitious" or "ideal" concepts, for the purpose of simplifying discoveries and shortening the arguments, like imaginary roots in ordinary analysis. Finally, he indicates one more field of ideas by means of which he attempts to justify the legitimacy of his conclusions—this was the "principle of continuity" which has a connection with the passage to the limit. However, all attempts of Leibniz to justify his calculus were apparently not entirely convincing even to himself. In one of his memoirs, considering the question as to whether the infinitesimals do in fact exist and whether they can be justified rigorously, Leibniz stated: "I think that this can be regarded as doubtful."

On the other hand, in one of his polemic papers he says: "I greatly appreciate the assiduity of the people who attempt to prove everything including even the original concepts; however, I would not advise the hindrance, by exaggerated thoroughness, of the art of discovery, or on this pretence to disregard the best discoveries and deprive oneself of their results...:" Thus, having no conviction and unable to justify the calculus created by himself, Leibniz considered its application justified by the results to which it leads.

This state of affairs is well described by Marx in the following words concerning the mathematicians of that epoch: "They themselves believed in the mystical nature of the newly discovered calculus which yielded correct results (and striking results in geometric applications) by a mathematically incorrect method. Thus they mystified themselves and thus valued the new discovery even more...."†

233. Postscript. The subsequent century was marked by a further development of mathematical analysis, its methods were perfected and its field of application was considerably enlarged. Nevertheless, to a great extent it preserved its "mystical" nature: its foundations, which were often subject to criticism, remained vague.

It is true that the concept of limit outlined only by the mathematicians of the seventeenth century was subsequently made more precise.

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† K. Marx. *Mathematical Manuscripts* (Under the Badge of Marxism, 1, 1933), p. 65 (in Russian).

In the foreword to *Differential Calculus* (1755) the outstanding St. Petersburg Academician Leonhard Euler (1707–1783) clearly speaks of the limit which more and more closely approaches the ratio of the increments of two quantities as the increments themselves become smaller and smaller. We have already mentioned this fact in Sec. 26, but we also emphasized that in Euler's treatise itself the concept of limit is not used once. About the same time the French mathematician and philosopher Jacques le Lond d'Alembert (1717–1783) in his papers in the celebrated *Encyclopedia* stated that he was convinced that "the theory of limits is the foundation of the true metaphysics of the differential calculus". At the end of the eighteenth century the application of the theory of limits in analysis and geometry was extensively advocated by the Russian mathematician and physicist, Academician Semen Yemelyanovitch Guryev (1764–1813). Nevertheless, the concept of a limit did not, in fact, become the real weapon for creating the foundations of mathematical analysis. Thus in 1797, Lazare Carnot (1753–1823) announced his *Meditations on the Metaphysics of Infinitesimals* where, repeating the known conjecture, he attempts to justify the continuous correctness of the results deduced by means of doubtful methods by a mutual compensation of errors.

Only the mathematicians of the early part of the nineteenth century, especially Augustin Louis Cauchy (1789–1857), made the concept of limit the real foundation of a sound construction of mathematical analysis as a whole, thus finally eradicating any mysticism from it. Incidentally, as we know, this foundation still contained a gap—there was no rigorous justification of the concept of real numbers and no discovery of the continuity of the field of real numbers; this was only accomplished in the second half of the last century.

We hope that the reader has been able to see the whole picture of the origins and creation of the fundamental concepts—of the differential and integral calculus as investigated in the present volume.

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